Two-Stage Majoritarian Choice

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Abstract

We propose a class of decisive collective choice rules that rely on an exogenous linear ordering to partition the majority relation into two acyclic relations. The first relation is used to make a shortlist of the feasible alternatives before the second is used to make a final choice.

Rules in this class are characterized by four properties: two classical rationality requirements (Sen's *expansion consistency* and Manzini and Mariotti's *weak WARP*); and adaptations of two natural collective choice conditions (Arrow's *independence of irrelevant alternatives* and Saari and Barney's *no preference reversal bias*). These rules also satisfy a number of other desirable properties including a version of May's *positive responsiveness.*

JEL Classification: D71, D72.

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1 Introduction

In many collective choice settings, rules that recommend more than one alternative are inappropriate. When it comes to selecting a political leader or a public policy, for instance, it is essential to be decisive. May (1952) shows that majority voting is the only reasonable way to decide between two alternatives.[1] With more alternatives, no rule that is faithful to the majority opinion can choose rationally. The root of the problem is the well-known Condorcet (1785) paradox: the majority relation may involve cycles. Arrow (1951) shows that the problem persists even with rules that are not majoritarian: barring dictatorship, there is no way to make rational and Pareto-efficient choices that satisfy the *independence of irrelevant alternatives* (IIA). We take Arrow's result as

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[1] In the sequel, we assume that the majority relation is decisive. This assumption is fairly innocuous for large electorates; and it is automatically satisfied when voter preferences are strict and the number of voters is odd.

good reason not to give up on majority voting. The goal, as we see it, is to design collective choice rules that are decisive, faithful to the majority view, and as rational as possible.

We propose a class of rules that not only meet these objectives but also exhibit a number of other desirable features—including versions of Arrow's *IIA*, May's *positive responsiveness* and Saari and Barney's (2003) *no preference reversal bias*. Not least among the virtues of these rules is their simplicity. Each uses a linear ordering to partition the majority relation into two acyclic relations. Then, as in Manzini and Mariotti's (2007) *rational shortlist methods*, the first relation is used to pare down the set of feasible alternatives before the second is used to make a final choice. While the linear orderings used by our rules are exogenous in principle, the choice setting itself often suggests a way to order the alternatives.

2 The problem

Given a finite universe of social alternatives X, let $\mathcal{X} = \{A \in 2^X \mid 2 \leq |A|\}$ denote the set of *agendas* and $\mathcal{T}$ the set of *tournaments* on X.[2] We interpret each tournament $T \in \mathcal{T}$ to be the majority relation induced by an underlying profile of agent preferences over X (McGarvey, 1953). Given a tournament T and an agenda A, the problem is to recommend one alternative in A.

Our object of interest is a *choice rule*, that is a mapping $f : \mathcal{T} \times \mathcal{X} \to X$ such that $f(T; A) \in A$ for each $T \in \mathcal{T}$ and $A \in \mathcal{X}$. For each tournament $T \in \mathcal{T}$, $f(T; \cdot) : \mathcal{X} \to X$ defines a *choice function*. We require that our choice rules be faithful to each tournament $T \in \mathcal{T}$ in the following sense:

Faithfulness. *For all $T \in \mathcal{T}$ and $a, b \in X$: aTb implies $f(T; \{a, b\}) = a$.*

To put it differently, we require binary choices to be consistent with majority rule.

Given a binary relation R on X, let $\max(R; A) := \{a \in A \mid \nexists b \in A : bRa\}$ denote the set of maximal elements of R in A. (If $\max(R; A)$ is a singleton, then we simplify by writing $\max(R; A) = a$ instead of $\max(R; A) = \{a\}$.) Let $\mathcal{P}$ denote the set of linear orderings on X.[3] A choice function $f(T; \cdot)$ is *rational* if there is some linear ordering $P \in \mathcal{P}$ such that $f(T; A) = \max(P; A)$ for all agendas $A \in \mathcal{X}$. If f is faithful, then $f(T; \cdot)$ cannot be rational unless T is a linear ordering. The question is whether there are faithful choice rules for which the choice function $f(T; \cdot)$ is rational when the tournament T is a linear ordering and not *too* irrational otherwise.

Some of the simplest faithful choice rules rely on an exogenous linear ordering $P \in \mathcal{P}$. The idea is to give an edge to alternatives ranked higher by P and thus guarantee a single-valued choice when the alternatives are difficult to distinguish (as in a Condorcet cycle $3T2T1T3$).

One natural approach uses P as a tie-breaking device to make a selection from a *Condorcet-consistent choice correspondence*, that is a mapping $F : \mathcal{T} \times \mathcal{X} \to \mathcal{X}$ such that, for all $T \in \mathcal{T}$ and $A \in \mathcal{X}$: (i) $F(T; A) \subseteq A$; and (ii) $F(T; A) = \{a\}$ if aTb for all $b \in A \setminus \{a\}$. Formally, the choice

[2]A tournament T is an *asymmetric* ($\nexists a, b : aTb$ and bTa) and *total* ($\forall a, b : aTb, bTa$, or $a = b$) binary relation.
[3]A linear ordering P is an asymmetric, total and *transitive* ($\forall a, b, c : aPbPc \Rightarrow aPc$) binary relation.

rule F_P generated by the choice correspondence F and the tie-breaking device $P \in \mathcal{P}$ is defined, for all $T \in \mathcal{T}$ and $A \in \mathcal{X}$, by $F_P(T; A) := \max(P; F(T; A))$.

Another approach uses P to define a succession of binary elimination votes that ultimately determine the choice from each agenda. For any agenda $A = \{a_1, ..., a_m\} \in \mathcal{X}$, label the alternatives so that $a_1 P ... P a_m$. Then, define $w_0(T; A) := a_m$ and, for $k = 1, ..., m-1$, recursively define

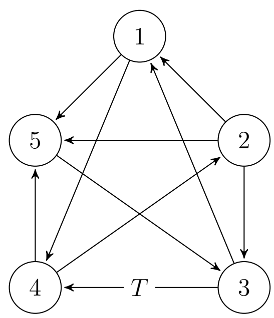
$$w_k(T;A) := \begin{cases} w_{k-1}(T;A) & \text{if } w_{k-1}(T;A) T a_{m-k}, \\ a_{m-k} & \text{otherwise.} \end{cases}$$

The first vote eliminates a_m or a_{m-1}. At any subsequent vote, the winner $w_{k-1}(T; A)$ from the previous vote is put up against the next alternative a_{m-k} from the list. The *successive elimination* rule s_P induced by $P \in \mathcal{P}$ is defined, for all $T \in \mathcal{T}$ and $A \in \mathcal{X}$, by $s_P(T; A) := w_{m-1}(T; A)$.

Both of these approaches lead to choice rules that lack in basic features of rationality:

Example 1 (Selection from the uncovered set). *The* uncovered set *choice correspondence $UC : \mathcal{T} \times \mathcal{X} \to \mathcal{X}$ (Landau, 1951; Fishburn, 1977; Miller, 1977) is defined, for all $T \in \mathcal{T}$ and $A \in \mathcal{X}$, by $UC(T; A) := \{a \in A \mid \forall b \in A \setminus \{a\} : \text{(i) } aTb \text{ or (ii) } aTcTb \text{ for some } c \in A\}$.*

Clearly, UC is Condorcet-consistent. For $X = \{1, 2, 3, 4, 5\}$, consider the tournament T below:



For the linear ordering $P = 1, ..., 5$ (with the alternatives listed in decreasing order of P):

$$UC_P(T; \{1,2,3,4\}) = 2 = UC_P(T; \{2,5\}) \text{ but } UC_P(T; \{1,2,3,4,5\}) = 1.$$

Thus, alternative 2 is chosen from $\{1, 2, 3, 4\}$ and $\{2, 5\}$ but not their union.[4] *Moreover,*

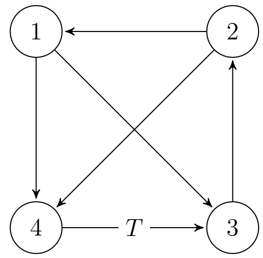
$$UC_P(T; \{1,2\}) = 2 = UC_P(T; \{1,2,3,4\}) \text{ but } UC_P(T; \{1,2,4\}) = 1.$$

So, 2 is chosen over 1 from $\{1, 2\}$ and $\{1, 2, 3, 4\}$ but not the intermediate agenda $\{1, 2, 4\}$.[5]

Example 2 (Successive elimination). *For $X = \{1, 2, 3, 4\}$, consider the tournament T below:*

[4] The same choice pattern can also arise if we start with the *top cycle* correspondence TC (as defined in Section 4 below). If we modify T so that $4T'1$, $TC_P(T'; \{1,2,3,4\}) = 2 = TC_P(T'; \{2,5\})$ but $TC_P(T'; \{1,2,3,4,5\}) = 1$.

[5] To see that this choice pattern cannot arise if we start with TC, suppose $TC_P(T; A) = a = TC_P(T; \{a, b\})$ and $TC_P(T; B) = b$ for $\{a, b\} \subseteq B \subseteq A$. Since $TC_P(T; \{a, b\}) = a$ and $TC_P(T; B) = b$, bPa. Since $a \in TC(T; A)$ and $b = c_1 T ... T c_n = a$ for some $c_1, ..., c_n \in B$, $b \in TC(T; A)$. Since bPa, this contradicts $TC_P(T; A) = a$.



For the successive elimination procedure induced by the linear ordering $P = 1, ..., 4$:

$$s_P(T;\{1,4\}) = s_P(T;\{1,2,3\}) = 1 \text{ but } s_P(T;\{1,2,3,4\}) = 2.$$

So, 1 is chosen from the agendas $\{1,4\}$ and $\{1,2,3\}$ but not their union. Moreover,

$$s_P(T;\{1,2\}) = s_P(T;\{1,2,3,4\}) = 2 \text{ but } s_P(T;\{1,2,3\}) = 1.$$

Thus, 2 is chosen over 1 from $\{1,2\}$ and $\{1,2,3,4\}$ but not the intermediate agenda $\{1,2,3\}$.[6]

The two rationality properties violated by the choice rules from Examples 1 and 2 are:

Expansion Consistency. *For all $T \in \mathcal{T}$, $a \in X$, and $A, B \in \mathcal{X}$:*

$$f(T;A) = a = f(T;B) \text{ implies } f(T;A \cup B) = a.$$

Weak WARP. *For all $T \in \mathcal{T}$, distinct $a, b \in X$, and $A, B \in \mathcal{X}$ such that $\{a,b\} \subseteq B \subseteq A$:*

$$f(T;\{a,b\}) = a = f(T;A) \text{ implies } f(T;B) \neq b.$$

Expansion Consistency dates back to Sen (1971). Weak WARP was first introduced by Manzini and Mariotti (2007) and later studied more extensively by Cherepanov et al. (2013). Both properties weaken Samuelson's (1938) *weak axiom of revealed preference* (WARP), which requires $f(T;B) = a$ if $f(T;A) = a$ and $a \in B \subseteq A$. Since WARP characterizes rational choice in our setting, it is incompatible with the requirement that f satisfies Faithfulness.

3 Two-stage majoritarian rules

We propose a class of choice rules that satisfy Faithfulness, Expansion Consistency and Weak WARP. Like the rules from Examples 1 and 2, each relies on an exogenous linear ordering $P \in \mathcal{P}$. For our rules, the role of P is to partition the given tournament $T \in \mathcal{T}$ into two acyclic binary

[6] The same choice patterns arise under the *amendment procedure* a_P (Miller, p. 779; Moulin, 1986, p. 287). Following our convention (that higher-ranked alternatives in P are more privileged), the linear ordering $P = 1, 2, 3, 4$ corresponds to the tree $\Gamma_4(4, 3, 2, 1)$ in Moulin. For the tournament T given in Example 2, the corresponding choice function gives $a_P(T;A) = s_P(T;A)$ for all $A \in \mathcal{X}$.

relations $T \cap P$ and $T \setminus P$. The first of these relations is then used to obtain a preliminary shortlist of the feasible alternatives in $A \in \mathcal{X}$ while the second to make a final choice. Formally, the *two-stage majoritarian choice rule* f_P *based on* $P \in \mathcal{P}$ is defined, for all $T \in \mathcal{T}$ and $A \in \mathcal{X}$, by

$$f_P(T; A) := \max(T \setminus P; \max(T \cap P; A)). \tag{1}$$

For each tournament $T \in \mathcal{T}$, the choice function $f_P(T; \cdot)$ defines a *rational shortlist method* in the sense of Manzini and Mariotti (2007).[7] Formally, a choice function $c : \mathcal{X} \to X$ is a rational shortlist method if there is a pair of asymmetric binary relations (P_1, P_2) (called *rationales*) on X such that $c(A) = \max(P_2; \max(P_1; A))$ for all $A \in \mathcal{X}$. To ensure that choice is single-valued, Manzini and Mariotti's model imposes non-trivial restrictions on the rationales (Lemma 2 of Dutta and Horan, 2015). When the rationales are built by splitting the tournament T into acyclic relations using a linear ordering P, these restrictions are satisfied *regardless* of T or P.

To see this, fix an agenda $A \in \mathcal{X}$. Since the binary relation $T \cap P$ is acyclic, the shortlist $M_A := \max(T \cap P; A)$ must be nonempty. The single-valuedness of $\max(T \setminus P; M_A)$ then follows from the acyclicity and totality of the binary relation $T \setminus P$ on M_A.

This argument only requires $T \cap P$ and $T \setminus P$ to be acyclic. Since P is also total, more can be said. Given a binary relation R on X, let $R^{-1} := \{(a, b) \in X^2 \mid (b, a) \in R\}$ denote the *reversal* of R. Then, $f_P(T; A) = \max(T \setminus P; M_A) = \max(T \cap P^{-1}; M_A) = \max(P^{-1}; \max(T \cap P; A))$. If we interpret aPb as a being "higher-ranked" than b, then $f_P(T; \cdot)$ may be understood to choose the lowest-ranked alternative that defeats all higher-ranked alternatives by majority. To illustrate:

Example 3 (Two-stage majoritarian rules). *For $P = 1, ..., 4$, the tournament T from Example 2 gives rationales $P_1 = T \cap P = \{(1, 3), (1, 4), (2, 4)\}$ and $P_2 = T \setminus P = \{(2, 1), (3, 2), (4, 3)\}$.*

To illustrate the resulting two-stage majoritarian rule f_P, first consider the Condorcet cycle $A = \{1, 2, 3\}$. Since $1P_13$, alternative 3 is eliminated in the first stage, leaving the shortlist $\{1, 2\}$. Since $2P_21$, alternative 1 is eliminated in the second stage, giving the final choice $f_P(T; A) = 2$.

Letting $f_P^{-1}(T; x) := \{A \in \mathcal{X} \mid f(T; A) = x\}$, the same kind of reasoning establishes that:

$$\begin{aligned}
f_P^{-1}(T; 1) &= \{\{1, 3\}, \{1, 4\}\}, \\
f_P^{-1}(T; 2) &= \{\{2, 1\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}, \\
f_P^{-1}(T; 3) &= \{\{2, 3\}, \{2, 3, 4\}\}, \text{ and} \\
f_P^{-1}(T; 4) &= \{\{3, 4\}\}.
\end{aligned}$$

By definition, every two-stage majoritarian rule f_P satisfies Faithfulness. Since the choice function $f_P(T; \cdot)$ is a rational shortlist method for each $T \in \mathcal{T}$, Manzini and Mariotti's characterization implies that f_P also satisfies Expansion Consistency and Weak WARP.

[7] More specifically, it defines a rational shortlist method where both rationales are acyclic. For an axiomatization of this model in the setting of individual choice, see Houy (2008).

These same properties are satisfied by some other choice rules (such as the rules described in Example 4 below). What distinguishes two-stage majoritarian rules is the kind of consistency that it imposes *across* tournaments. This consistency can be factored into two requirements.

The first is a natural adaptation of Arrow's IIA to our setting (originally formulated by Moulin, 1986, p. 278). Let $T|_A$ denote the restriction of the tournament $T \in \mathcal{T}$ to the agenda $A \in \mathcal{X}$.

Choice IIA. *For all $T, T' \in \mathcal{T}$ and $A \in \mathcal{X}$ such that $T|_A = T'|_A$: $f(T;A) = f(T';A)$.*

To paraphrase, the majority view of infeasible alternatives cannot affect choice. Not only is this property satisfied by two-stage majoritarian rules, it is also satisfied by the rules from Examples 1 and 2 (as well as the variations of these rules discussed in footnotes 4 and 6).

The second stipulates that choice must improve when all majority comparisons are reversed. (Recall that $T^{-1} := \{(a,b) \in X^2 \mid (b,a) \in T\}$ denotes the reversal of a tournament $T \in \mathcal{T}$.)

Reversal Improvement. *For all $T \in \mathcal{T}$ and $A \in \mathcal{X}$: $f(T;A)Tf(T^{-1};A)$.*

This property strengthens Faithfulness, which coincides with the special case where $|A| = 2$. It also strengthens a condition that Saari and Barney (2003, p. 17) originally proposed for the richer setting where choice rules may depend on individual preferences. Their property requires the collective choice to change when the all individual preferences are reversed. In our setting, this amounts to the requirement that $f(T;A) \neq f(T^{-1};A)$. Reversal Improvement further stipulates that reversing the majority preference must *improve* choice. The rules from Examples 1 and 2 (as well as the variations discussed in footnotes 4 and 6) all violate this requirement.

What motivates us to strengthen Saari and Barney's condition in this way is the conceit that changes to the majority opinion should impact choice for the better. This makes our Reversal Improvement condition similar, in spirit at least, to May's positive responsiveness (which is discussed at greater length in Section 4.2 below). The main difference is that May's condition relates to changes that reinforce the support for a particular choice. Our condition instead relates to changes that reverse all comparisons that led to a particular choice.

In combination with Expansion Consistency and Weak WARP, Choice IIA and Reversal Improvement characterize two-stage majoritarian rules. To state our result formally:

Theorem. *A choice rule $f : \mathcal{T} \times \mathcal{X} \to X$ is a two-stage majoritarian choice rule if and only if it satisfies Expansion Consistency, Weak WARP, Choice IIA and Reversal Improvement.*

Proof. It is straightforward to check the necessity of the axioms.

Since sufficiency is immediate if $|X| = 2$, suppose $|X| \geq 3$. Define the binary relation R on X such that, for all $x, y \in X$: xRy if there is some $T \in \mathcal{T}$ and $z \in X$ such that $xTzTyTx$ and $f(T;\{x,y,z\}) = x$. Equivalently, by Reversal Improvement, xRy if there is some $T' \in \mathcal{T}$ and $z \in X$ such that $xT'yT'zT'x$ and $f(T';\{x,y,z\}) = z$. We write xIy if neither xRy nor yRx.

Step 1. *R is (i) asymmetric and (ii) transitive.*

(i) To the contrary, suppose $xRyRx$ for some $x, y \in X$. By definition, there are $c, d \in X \setminus \{x, y\}$ and $T, T' \in \mathcal{T}$ such that $xTyTcTx$, $xT'yT'dT'x$, $f(T; \{x, y, c\}) = c$, and $f(T'; \{x, y, d\}) = y$. By Choice IIA, it follows that $c \neq d$. For $|X| = 3$, this yields a contradiction directly. For $|X| \geq 4$, consider $T^* \in \mathcal{T}$ such that $T^*|_C = T|_C$ for $C := \{x, y, c\}$, $T^*|_D = T'|_D$ for $D := \{x, y, d\}$, and cT^*d. By Faithfulness, $f(T^*; \{c, d\}) = c$ and $f(T^*; \{y, d\}) = y$. Since $f(T^*; C) = c$ and $f(T^*; D) = y$ by Choice IIA, Expansion Consistency leads to the following contradiction:

$$c = f(T^*; C \cup \{c, d\}) = f(T^*; \{x, y, c, d\}) = f(T^*; D \cup \{c, d\}) = d.$$

(ii) Suppose $xRyRz$. Consider $T \in \mathcal{T}$ such that $xTyTzTx$. If $f(T; \{x, y, z\}) \neq x$, yRx or zRy. Since $xRyRz$, this contradicts the asymmetry of R. So, $f(T; \{x, y, z\}) = x$ and xRz. □

Step 2. *There are exactly two distinct $a, b \in X$ such that aIb; and a, bRc for all $c \in X \setminus \{a, b\}$.*

First, suppose there are two pairs of distinct alternatives $\{x, y\}$ and $\{z, w\}$ (with $x \neq z, w$ and $z \neq x, y$) such that xIy and zIw. First, consider $T \in \mathcal{T}$ such that $xTzTwTx$. Since zIw, the definition of R implies zRx and wRx. Next, consider $T \in \mathcal{T}$ such that $xTyTzTx$. Then, xRz since xIy. Since $xRzRx$ contradicts the asymmetry of R, aIb for at most one pair $\{a, b\}$.

If there is no such pair, then R is a linear ordering by Step 1. Suppose aRb and bRc for all $c \in X \setminus \{a, b\}$. Then, by definition of R, there is some $d \in X \setminus \{a, b\}$ such that dRb. Since this is a contradiction, there must be exactly one pair $\{a, b\}$ such that aIb. Finally, from the definition of R, it follows that aRc and bRc for all $c \in X \setminus \{a, b\}$. □

Complete R into a linear ordering P by defining aPb and xPy if xRy for $x, y \in X$.

Step 3. *For all $T \in \mathcal{T}$ and $A \in \mathcal{X}$, $f(T; A) = f_P(T; A)$.*

The proof is by strong induction on $|A|$. For the base cases $|A| = 2, 3$, the result follows from Reversal Improvement, Expansion Consistency, and the definition of R. For the induction step, suppose that the result holds for $2 \leq |A| < n$ and consider the case $|A| = n \geq 4$.

By the induction hypothesis, it suffices to show that $f(T; A) = f_P(T; A)$ for all $T \in \mathcal{T}$. Labelling the alternatives of $A = \{a_1, ..., a_n\}$ so that $a_1 P ... P a_n$, this is equivalent to showing that

$$f(T; A) = \begin{cases} a_n & \text{if } a_n = \max(T; A), \\ f(T; A \setminus \{a_n\}) & \text{otherwise.} \end{cases} \tag{2}$$

First, suppose $f(T; A \setminus \{a_n\}) = x$ and recursively define a sequence $\langle b_i \rangle_{i=0}^m$ in A such that:

(i) $b_0 := a_n$;
(ii) $B_{i+1} := \{y \in A \mid y(P \cap T)b_i\}$ and $b_{i+1} := \max(P; B_{i+1})$; and,
(iii) m is the smallest index such that $b_m P x$, $b_m = x$, or $B_{m+1} = \emptyset$.

Next, define $B := \{b_0, ..., b_m\}$. Since $f(T; A \setminus \{a_n\}) = x$, there are two possibilities. If $a_n T a_i$ for all $i = 1, ..., n-1$, then $B = \{a_n\} = \{b_0\}$. Otherwise, $B = \{b_0, ..., b_m\} \neq \{b_0\}$ with the features that: (a) $b_m(T \cap P)...(T \cap P)b_0$; and (b) $x(T \setminus P)b_m$ or $x = b_m$. This leaves three cases:

Case 1. If $B = \{a_n\}$, then $a_n = \max(T; A)$. By the induction hypothesis and the application of Expansion Consistency to $f(T; \{a_i, a_n\}) = a_n$ for $i = 1, ..., n-1$, it follows that $f(T; A) = a_n$.

Case 2. If $1 < |B \cup \{x\}| < n$, then $f(T; B \cup \{x\}) = x$ by (a)-(b) above and the induction hypothesis. So, $f(T; A) = f(T; (B \cup \{x\}) \cup (A \setminus \{a_n\})) = x$ by Expansion Consistency.

Case 3. If $B \cup \{x\} = A$, then $x \in \{a_1, a_2\}$ by (a)-(b) above. What is more, the definition of B implies: (c) $a_{i-1}Ta_i$ for $i = 4, ..., n$; and (d) a_iTa_j for $i = 4, ..., n$ and all $j < i-1$.

First, suppose $x = a_1$. By definition of B, $a_1Ta_2Ta_3Ta_1$. Given (c)-(d), $a_{i-2}Ta_{i-1}Ta_iTa_{i-2}$ for $i = 3, ..., n$. So, $f(T; \{a_{i-2}, a_{i-1}, a_i\}) = a_{i-2}$ by the induction hypothesis. Since $f(T; \{a_{i-2}, a_i\}) = a_i$ by the induction hypothesis, Weak WARP precludes $f(T; A) = a_i$. So, $f(T; A) \in \{a_1, a_2\}$.

To rule out $f(T; A) = a_2$, consider the reversal T^{-1} of T. Since $a_{i-2}T^{-1}a_iT^{-1}a_{i-1}T^{-1}a_{i-2}$ for $i = 3, ..., n$, the same kind of argument given for T implies $f(T^{-1}; A) \in \{a_1, a_2\}$. Since $f(T; A) \in \{a_1, a_2\}$ and a_1Ta_2, Reversal Improvement then implies $f(T; A) = a_1 = x$.

Next, suppose $x = a_2$. Then, from the definition of B and the fact that $f(T; (A \setminus \{a_n\}) = a_2$, it follows that a_1Ta_3 and a_2Ta_1. We distinguish two possibilities: (i) a_3Ta_2; and (ii) a_2Ta_3.

(i) Given $a_2Ta_1Ta_3Ta_2$ and (c), the same kind of argument used for $x = a_1$ establishes $f(T; A) = a_2$. (The difference is that $f(T; A) = a_3$ is ruled out by $a_2Ta_1Ta_3Ta_2$ while $f(T; A) = a_4$ is ruled out by $a_1Ta_3Ta_4Ta_1$. In turn, $f(T^{-1}; A) = a_3$ is ruled out by $a_2T^{-1}a_3T^{-1}a_1T^{-1}a_2$ while $f(T^{-1}; A) = a_4$ is ruled out by $a_1T^{-1}a_4T^{-1}a_3T^{-1}a_1$.)

(ii) By the induction hypothesis: $f(T; A \setminus \{a_1\}) = a_2$ given a_2Ta_3 and (c); and $f(T; \{a_1, a_2\}) = a_2$ given a_2Ta_1. So, $f(T; A) = f(T; (A \setminus \{a_1\}) \cup \{a_1, a_2\}) = a_2 = x$ by Expansion Consistency. ■

Remark. Our proof implies that a choice rule satisfying the axioms can be represented with exactly two linear orderings (denoted P_a and P_b below) that differ only in terms of how they rank the top two alternatives a and b identified in Step 2 of the proof. This non-uniqueness is an inherent feature of the model. Observe that formula (2) implies $f_{P_a}(T; \{a, b\}) = \max(T; \{a, b\}) = f_{P_b}(T; \{a, b\})$ for all $T \in \mathcal{T}$. As a result, $f_{P_a}(T; A) = f_{P_b}(T; A)$ for all $T \in \mathcal{T}$ and $A \in \mathcal{X}$.

4 Further remarks

4.1 Flexibility and Pareto sub-optimality

The *top cycle* choice correspondence $TC : \mathcal{T} \times \mathcal{X} \to \mathcal{X}$ (Good, 1971; Schwartz, 1972; Smith, 1973) is defined by $TC(T; A) := \{a \in A \mid \forall b \in A \setminus \{a\} : a = c_1T...Tc_n = b \text{ for some } c_1, ..., c_n \in A\}$. Like the uncovered set choice correspondence, TC is Condorcet-consistent.

For all $T \in \mathcal{T}$ and $A \in \mathcal{X}$, the set of alternatives chosen by some two-stage majoritarian rule coincides with the top cycle, that is $TC(T; A) = \{f_P(T; A) \mid P \in \mathcal{P}\}$. To see this in one direction, note that $f_P(T; A) = f_P(T; TC(T; A))$. In the other, fix a path $a = a_1 T ... T a_m$ from $a \in TC(T; A)$ that covers $TC(T; A)$. Then, $f_P(T; A) = a$ for any linear ordering $P \in \mathcal{P}$ such that $a_m P ... P a_1$. Since $TC(T; A) = \{s_P(T; A) \mid P \in \mathcal{P}\}$ as well (Miller, 1977), two-stage majoritarian rules provide the same *flexibility* to the designer as successive elimination rules (Example 2).

It is well known that, for some tournaments $T \in \mathcal{T}$ and agendas $A \in \mathcal{X}$ such that $|A| \geq 4$, the top cycle $TC(T; A)$ may contain alternatives that are Pareto dominated at preference profiles consistent with T (for an example, see Moulin, 1986, p. 274). Given their flexibility, this means that all two-stage majoritarian rules sometimes make Pareto sub-optimal choices.

4.2 The connection to May

Two-stage majoritarian rules satisfy a natural adaptation of May's *positive responsiveness* to the tournament setting. To state this adaptation (originally formulated by Moulin, 1986, p. 285), say that a binary relation $R^{\uparrow a}$ on X *improves* an alternative $a \in X$ relative to another binary relation R on X if, for all $x, y \in X \setminus \{a\}$: (i) $aRx \Rightarrow aR^{\uparrow a}x$; and (ii) $xRy \Leftrightarrow xR^{\uparrow a}y$.

T-**Monotonicity.** *For all $T \in \mathcal{T}$, $a \in X$, $T^{\uparrow a} \in \mathcal{T}$ that improves a, and $A \in \mathcal{X}$:*

$$f(T; A) = a \text{ implies } f(T^{\uparrow a}; A) = a.$$

In words: improving the majority view of a chosen alternative can only reinforce its choice.

To see that two-stage majoritarian rules satisfy this property, recall that $f_P(T; A)$ is the lowest-ranked alternative in A that defeats all higher-ranked alternatives by majority. Improving $f_P(T; A)$ relative to T cannot change this: $f_P(T; A)$ still defeats all higher-ranked alternatives; and every alternative ranked below $f_P(T; A)$ is still defeated by some higher-ranked alternative.

It is well understood that the rules from Example 2 (as well as the rules from footnotes 4 and 6) satisfy T-Monotonicity.[8] However, the rules from Example 1 do not. To illustrate, consider $X = \{1, 2, 3, 4\}$ and the ordering $P = 4, 3, 2, 1$. Then, $UC_P(T; X) = 3$ for the tournament T from Example 2 while $UC_P(T'; X) = 4$ for the tournament T' that improves 3 relative to 1.

4.3 The role of P

A minor variation on the argument used to establish that two-stage majoritarian rules satisfy T-Monotonicity shows that every f_P is also monotonic with respect to the linear ordering P that defines it. In other words, two-stage majoritarian rules satisfy the following property:

[8]See Exercise 9.4(c) of Moulin (1988, p. 250) for s_P and the Corollary to Theorem 9.5 (p. 247) for a_P. Horan (2020) shows that a much broader range of binary trees (which he calls "simple agendas") have the same feature. For TC_P, simply note that the top cycle cannot gain new members by improving one of its members.

P-Monotonicity. *For all $P \in \mathcal{P}$, $a \in X$, $P^{\uparrow a} \in \mathcal{P}$ that improves a, $T \in \mathcal{T}$, and $A \in \mathcal{X}$:*

$$f_P(T; A) = a \text{ implies } f_{P^{\uparrow a}}(T; A) = a.$$

This property captures the idea that alternatives ranked higher by P are more privileged.

The rules from Examples 1 and 2 (as well as the related rules from footnotes 4 and 6) satisfy an analogous monotonicity property.[9] The difference is that the linear ordering P plays a less intrusive role for two-stage majoritarian rules than for these other rules. To see this, consider the successive elimination rules from Example 2. The alternative $s_P(T; A)$ must defeat all higher-ranked alternatives in any agenda $A \in \mathcal{X}$. Because the same is true for $f_P(T; A)$, this means that, when the alternatives $s_P(T; A)$ and $f_P(T; A)$ differ, $f_P(T; A)$ must be ranked lower in terms of P and, thus, preferred by a majority. By the same reasoning, $f_P(T; A)$ must be weakly preferred by majority to the alternatives $TC_P(T; A)$ and $a_P(T; A)$ chosen by the top cycle selection rule (footnote 4) and the amendment rule (footnote 6). Indeed, the same is true for the uncovered set selection rule from Example 1 when differences in flexibility are taken into account: if $f_P(T; A) \in UC(T; A)$, then $f_P(T; A)$ is weakly preferred by a majority to $UC_P(T; A)$.

Having said this, the choice of the linear ordering P still has a significant impact on the outcomes associated with the rule f_P. Fortunately, there is a natural way to define P in many collective choice settings. In the public policy setting, for instance, it is natural to order competing policies by increasing cost or decreasing equity. In legislative settings, it may be more natural to follow the convention of ordering proposals either by the time at which that they were tabled or by their degree of divergence from the *status quo* legislation (Rasch, 2000, p. 15). Finally, in the committee setting, it seems reasonable to use the preference of the chair, which is conventionally used as a tie-breaking device in committee decision-making (Robert, 2011, p. 405).

4.4 An extension

There may be scenarios where it is desirable to select the same alternative both for a tournament and its reversal. To accommodate this possibility, it is necessary to weaken the conclusion of Reversal Improvement to allow $f(T; A) = f(T^{-1}; A)$ as well as $f(T; A) T f(T^{-1}; A)$. Unlike Reversal Improvement, this *Weak Reversal Improvement* property does not imply Faithfulness.

In combination with Faithfuness and the other requirements in our Theorem, Weak Reversal Improvement defines a much broader class of choice rules. The next example describes a set of rules in this class that share the same basic structure as two-stage majoritarian rules.

Example 4. (General two-stage majoritarian rules) *Let $\mathcal{R}_2$ denote the set of weak orderings*[10] *R on X such that, for any $x \in X$, the indifference class $I_R(x) := \{y \in X \mid xRyRx\}$ contains*

[9] For UC_P and TC_P, the claim is straightforward. For s_P and a_P, see Exercise 9.5 of Moulin (1988, p. 250). A much broader class of binary trees introduced by Horan (2020) (called "priority agendas") have the same feature.

[10] A weak ordering R is a complete ($\forall a, b : aRb$ or bRa) and transitive binary relation.

at most two alternatives. For a weak ordering $R \in \mathcal{R}_2$, let g_R be the choice rule defined, for all $T \in \mathcal{T}$ and $A \in \mathcal{X}$, by

$$g_R(T; A) := \max(T \setminus R; \max(T \cap R; A)). \tag{3}$$

To see that g_R actually defines a choice rule, consider the linear ordering R_T that results from taking the lexicographic composition of R with a tournament $T \in \mathcal{T}$. Then, $g_R(T; A) = f_{R_T}(T; A)$ for all $A \in \mathcal{X}$. Not only does this show that g_R is well-defined, it shows that g_R is a two-stage majoritarian rule when R contains no indifferences (since, in that case, $R_T = R_{T'}$ for all $T, T' \in \mathcal{T}$). This is no longer true when $xRyRx$ for distinct $x, y \in X$. Then, x and y are compared by the first rationale $T \cap R$ regardless of $T \in \mathcal{T}$, something which cannot occur for a two-stage majoritarian rule.

Besides Faithfulness, Expansion Consistency, Weak WARP, Choice IIA, and Weak Reversal Improvement, g_R satisfies T-Monotonicity and (the analog of) P-Monotonicity for any weak ordering $R \in \mathcal{R}_2$. To see that g_R may violate Reversal Improvement when $|X| \geq 3$, consider a weak ordering $R \in \mathcal{R}_2$ such that $1R2R3R2$ and a tournament $T \in \mathcal{T}$ such that $1T2T3T1$. Then, by definition of g_R, it follows that $g_R(T; A) = 1 = g_R(T^{-1}; A)$ for the agenda $A = \{1, 2, 3\}$.

We close by noting that the rules from Example 4 provide precisely the same flexibility as two-stage majoritarian rules, that is $TC(T; A) = \{g_{\sigma R}(T; A) \mid \sigma \text{ is a permutation on } X\}$ for all $R \in \mathcal{R}_2$, $T \in \mathcal{T}$, and $A \in \mathcal{X}$. As in the case of two-stage majoritarian rules, the implication is that all of these rules sometimes make Pareto sub-optimal choices. This raises the question of whether an efficient choice rule is capable of satisfying all of the requirements listed in the last paragraph.

References

[1] Arrow, K.J. (1951), *Social choice and individual values.* Wiley: New York.

[2] Cherepanov, V., Feddersen, T., and Sandroni A. (2013). “Rationalization.” *Theoretical Economics* 8, 775–800.

[3] de Condorcet, N. (1785). “Essai sur l’application de l’analyse à la probabilité des décisions rendues à la pluralité des voix.”

[4] Dutta, R., and Horan, S. (2015). “Inferring rationales from choice: Identification for rational shortlist methods.” *American Economic Journal: Microeconomics* 7, 179–201.

[5] Fishburn, P.C. (1977). “Condorcet social choice functions.” *SIAM Journal on Applied Mathematics* 33, 469–489.

[6] Good, I. J. (1971). “A Note on Condorcet sets.” *Public Choice* 10, 97–101.

[7] Houy, N. (2008). “Progressive knowledge revealed preferences and sequential rationalizability.” École Polytechnique working paper.

[8] Horan, S. (2020). "Agendas in legislative decision-making." *Theoretical Economics*, forthcoming.

[9] Landau, H.G. (1951) "On dominance relations and the structure of animal societies: I. Effect of inherent characteristics." *Bulletin of Mathematical Biophysics* 13, 1–19.

[10] McGarvey, D. (1953). "A theorem on the construction of voting paradoxes." *Econometrica* 21, 608–610.

[11] Manzini, P. and Mariotti, M. (2007). "Sequentially rationalizable choice." *American Economic Review* 97, 1824–1839.

[12] May, K.O. (1952), "A set of independent necessary and sufficient conditions for simple majority decision." *Econometrica* 20, 680–684.

[13] Miller, N. (1977) "Graph-theoretical approaches to the theory of voting." *American Journal of Political Science* 21, 769–803.

[14] Moulin, H. (1986). "Choosing from a tournament." *Social Choice and Welfare* 3, 271–291.

[15] Moulin, H. (1988). *Axioms of cooperative decision making.* Economic Society Monographs No. 15, Cambridge University Press: Cambridge.

[16] Rasch, B.E. (2000). "Parliamentary floor voting procedures and agenda setting in Europe." *Legislative Studies Quarterly* 25, 3–23.

[17] Robert, H. (2011). *Robert's Rules of Order Newly Revised (11th edition).* Da Capo Press: Philadelphia.

[18] Saari, D.G., and S. Barney. (2003). "Consequences of reversing preferences." *The Mathematical Intelligencer* 25.4, 17–31.

[19] Samuelson, P. (1938). "A note on the pure theory of consumer's behavior." *Economica* 5, 61–71.

[20] Schwartz, T. (1972). "Rationality and the Myth of the Maximum." *Noûs* 6, 97–117.

[21] Sen, A. K. (1971). "Choice functions and revealed preference." *Review of Economic Studies* 38, 307–317.

[22] Smith, J. H. (1973). "Aggregation of preferences with variable electorate." *Econometrica* 41, 1027–1041.