Two-Stage Majoritarian Choice

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Abstract

We propose a class of decisive collective choice rules that rely on a linear ordering to partition the majority relation into two acyclic relations. The first relation is used to obtain a shortlist of the feasible alternatives while the second is used to make a final choice.

Rules in this class are characterized by four properties: two classical rationality requirements (Sen's *expansion consistency* and Manzini and Mariotti's *weak WARP*); and adaptations of two classical collective choice requirements (Arrow's *independence of irrelevant alternatives* and Saari and Barney's *no preference reversal bias*). These rules also satisfy some other desirable properties including a version of May's *positive responsiveness*.

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1 Introduction

In many collective choice settings, rules that recommend more than one alternative are inappropriate. When it comes to selecting a public policy or passing legislation, for instance, it is essential to be decisive. May (1952) shows that majority voting is the only reasonable way to decide between two alternatives.[1] With more alternatives, no rule that is faithful to the majority can always choose rationally. The root of the problem is the Condorcet (1785) paradox: the majority relation may involve cycles. Arrow (1951) shows that this problem extends to rules that are not majoritarian: barring dictatorship, there is no way to make rational and Pareto-efficient choices that satisfy the *independence of irrelevant alternatives* (IIA). We take Arrow's result as good reason not to give up on majority rule, but rather to search for collective choice rules that are decisive, faithful to the majority view, and as rational as possible.

We propose a class of rules that not only meet these objectives but also satisfy some additional desiderata—including versions of Arrow's *IIA*, May's *positive responsiveness* and Saari and Barney's (2003) *no preference reversal bias*. Not least among the virtues of these rules is their simplicity. Each relies on a linear ordering to partition the majority relation into two acyclic relations. Then, as in Manzini and Mariotti's (2007) *rational shortlist methods*, the first relation is used to pare down the set of feasible alternatives into a shortlist before the second relation is used to make a final choice. While the linear orderings used by the rules are in principle exogenous, the choice setting itself frequently suggests a natural way to order the alternatives.

2 The problem

Given a finite universe of social alternatives X, let $\mathcal{X} = \{A \in 2^X \mid 2 \leq |A|\}$ denote the set of *agendas* and $\mathcal{T}$ the set of *tournaments* on X.[2] We interpret each $T \in \mathcal{T}$ to be the majority relation induced by an underlying profile of agent preferences over X (McGarvey, 1953).

Given a tournament T and an agenda A, the problem is to recommend one alternative in A.[3] Formally, the object of interest is a *choice rule*, that is a mapping $f : \mathcal{T} \times \mathcal{X} \to X$ such that $f(T; A) \in A$ for each $T \in \mathcal{T}$ and $A \in \mathcal{X}$. We note that the restriction of a choice rule f to any tournament $T \in \mathcal{T}$ defines a classical *choice function* $f(T; \cdot) : \mathcal{X} \to X$.

We emphasize that a choice rule is *decisive* in that it must select a single alternative from each agenda. In our view, the idea of recommending a set of "acceptable" alternatives is problematic.

[1] In the sequel, we assume that the majority relation is decisive. This assumption is fairly innocuous for large electorates; and it is automatically satisfied when voter preferences are strict and the number of voters is odd.

[2] A tournament T is an *asymmetric* ($\nexists a, b : aTb$ and bTa) and *total* ($\forall a, b : aTb, bTa$, or $a = b$) binary relation.

[3] The general voting model takes the profile of individual preferences as the relevant input for social choice. While it is certainly restrictive to focus on rules that only rely on the associated majority relation, there is a rich tradition of voting rules called "tournament solutions" that does precisely this. (For a comprehensive treatment of tournament solutions, see Laslier, 1997.) Typically, these rules are justified on the basis of informational parsimony.

At best, it puts off what we view as the real task: to make a definite choice from the agenda. At worst, it delegates the task of choosing among the acceptable alternatives, a choice which is quite likely to have welfare implications,[4] to an *ad hoc* tie-breaking procedure.

As outlined, we focus on choice rules that are faithful to majority rule for binary choices:

Faithfulness. *For all $T \in \mathcal{T}$ and $a, b \in X$: aTb implies $f(T; \{a, b\}) = a$.*

For a binary relation R on X, let $\max(R; A) := \{a \in A \mid \nexists b \in A : bRa\}$ denote the set of maximal elements of R in $A \in \mathcal{X}$. (When this set is a singleton, we write $\max(R; A) = a$ instead of $\max(R; A) = \{a\}$.) Let $\mathcal{P}$ denote the set of linear orderings on X.[5] A choice function $f(T; \cdot)$ is *rational* if there is a linear ordering $P \in \mathcal{P}$ such that $f(T; A) = \max(P; A)$ for all $A \in \mathcal{X}$.

If f satisfies Faithfulness, then $f(T; \cdot)$ cannot be rational unless the tournament T is a linear ordering. The question is whether there are faithful choice rules for which $f(T; \cdot)$ is rational when T is a linear ordering and not *too* irrational otherwise. To gain broad legitimacy among the agents, it is essential for a rule to exhibit "consistency" in choice. Following in the Arrovian tradition, we seek to achieve greater consistency by limiting collective irrationality.

Some of the simplest faithful choice rules from the literature use an exogenous linear ordering $P \in \mathcal{P}$ to establish a *priority* among the alternatives. The basic idea is to give more of an "edge" to alternatives that are ranked higher by P and thus guarantee that choice is single-valued even when the alternatives are not easy to distinguish on principle (as in a Condorcet cycle).

One such approach uses the priority P as a tie-breaking device to make a selection from the set of alternatives recommended by a Condorcet-consistent choice correspondence. Formally, $F : \mathcal{T} \times \mathcal{X} \to 2^X$ is a *Condorcet-consistent correspondence* if, for all $T \in \mathcal{T}$ and $A \in \mathcal{X}$: (i) $F(T; A) \subseteq A$; and (ii) $F(T; A) = \{a\}$ if aTb for all $b \in A \setminus \{a\}$ (i.e., if a is the *Condorcet winner*). The choice rule F_P generated by the Condorcet-consistent correspondence F and the priority $P \in \mathcal{P}$ is defined, for all $T \in \mathcal{T}$ and $A \in \mathcal{X}$, by $F_P(T; A) := \max(P; F(T; A))$.

Another approach uses the priority P to define a succession of binary elimination votes. For any agenda $A = \{a_1, ..., a_m\} \in \mathcal{X}$, label the alternatives so that $a_1 P ... P a_m$. Then, define $w_0(T; A) := a_m$ and, for $k = 1, ..., m-1$, define

$$w_k(T; A) := \begin{cases} w_{k-1}(T; A) & \text{if } w_{k-1}(T; A) T a_{m-k}, \\ a_{m-k} & \text{otherwise.} \end{cases}$$

The first vote eliminates either a_m or a_{m-1}. At any subsequent vote, the winner $w_{k-1}(T; A)$ from the previous vote is paired against the next lowest priority alternative a_{m-k} that has not yet been put to a vote. The *successive elimination* rule s_P induced by the priority $P \in \mathcal{P}$ is then defined, for all $T \in \mathcal{T}$ and $A \in \mathcal{X}$, by $s_P(T; A) := w_{m-1}(T; A)$.

[4]If T is induced by a profile of strict preferences, for instance, no two alternatives are Pareto indifferent.

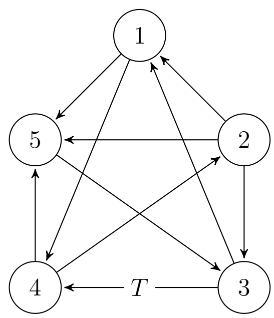
[5]A linear ordering P is an asymmetric, total and *transitive* ($\forall a, b, c : aPbPc \Rightarrow aPc$) binary relation.

Both of these approaches induce choice rules that lack basic features of rationality:

Example 1 (Selection from the uncovered set). *One well-known Condorcet-consistent correspondence is the* uncovered set *correspondence* $UC : \mathcal{T} \times \mathcal{X} \to \mathcal{X}$ *(Landau, 1951; Fishburn, 1977; Miller, 1980), which is defined, for all* $T \in \mathcal{T}$ *and* $A \in \mathcal{X}$*, by*

$$UC(T; A) := \{a \in A \mid \forall b \in A \setminus \{a\} : \text{(i) } aTb \text{ or (ii) } aTcTb \text{ for some } c \in A\}.$$

On the universe $X = \{1, 2, 3, 4, 5\}$*, consider the tournament* T *depicted below:*



For the priority $P = 1, ..., 5$ *(with the alternatives listed in decreasing order of* P*):*

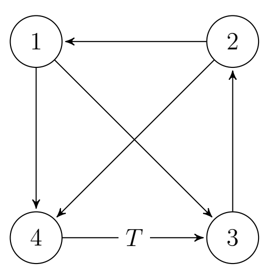
$$UC_P(T; \{1, 2, 3, 4\}) = 2 = UC_P(T; \{2, 5\}) \text{ but } UC_P(T; \{1, 2, 3, 4, 5\}) = 1.$$

Thus, alternative 2 *is chosen from* $\{1, 2, 3, 4\}$ *and* $\{2, 5\}$ *but not their union.*[6] *Moreover,*

$$UC_P(T; \{1, 2\}) = 2 = UC_P(T; \{1, 2, 3, 4\}) \text{ but } UC_P(T; \{1, 2, 4\}) = 1.$$

So, 2 *is chosen over* 1 *from* $\{1, 2\}$ *and* $\{1, 2, 3, 4\}$ *but not the intermediate agenda* $\{1, 2, 4\}$*.*[7]

Example 2 (Successive elimination). *For* $X = \{1, 2, 3, 4\}$*, consider the tournament* T *below:*



[6]The same choice pattern can also arise if we start with the *top cycle* correspondence TC (as defined in Section 4 below). If we modify T so that $4T'1$, $TC_P(T'; \{1, 2, 3, 4\}) = 2 = TC_P(T'; \{2, 5\})$ but $TC_P(T'; \{1, 2, 3, 4, 5\}) = 1$.

[7]To see that this choice pattern cannot arise if we start with TC, suppose $TC_P(T; A) = a = TC_P(T; \{a, b\})$ and $TC_P(T; B) = b$ for $\{a, b\} \subseteq B \subseteq A$. Since $TC_P(T; \{a, b\}) = a$ and $TC_P(T; B) = b$, bPa. Since $a \in TC(T; A)$ and $b = c_1T...Tc_n = a$ for some $c_1, ..., c_n \in B$, $b \in TC(T; A)$. Since bPa, this contradicts $TC_P(T; A) = a$.

For the successive elimination procedure induced by the priority $P = 1, ..., 4$:

$$s_P(T; \{1,4\}) = s_P(T; \{1,2,3\}) = 1 \text{ but } s_P(T; \{1,2,3,4\}) = 2.$$

So, 1 *is chosen from the agendas* $\{1,4\}$ *and* $\{1,2,3\}$ *but not their union. Moreover,*

$$s_P(T; \{1,2\}) = s_P(T; \{1,2,3,4\}) = 2 \text{ but } s_P(T; \{1,2,3\}) = 1.$$

Thus, 2 *is chosen over* 1 *from* $\{1,2\}$ *and* $\{1,2,3,4\}$ *but not the intermediate agenda* $\{1,2,3\}$.[8]

The choice rules from Examples 1 and 2 both violate the following rationality properties:

Expansion Consistency. *For all* $T \in \mathcal{T}$*,* $a \in X$*, and* $A, B \in \mathcal{X}$*:*

$$f(T; A) = a = f(T; B) \text{ implies } f(T; A \cup B) = a.$$

Weak WARP. *For all* $T \in \mathcal{T}$*, distinct* $a, b \in X$*, and* $A, B \in \mathcal{X}$ *such that* $\{a, b\} \subseteq B \subseteq A$*:*

$$f(T; \{a,b\}) = a = f(T; A) \text{ implies } f(T; B) \neq b.$$

Expansion Consistency dates back to Sen (1971). Weak WARP was first introduced by Manzini and Mariotti (2007) and later studied more extensively by Cherepanov et al. (2013). Both properties weaken Samuelson's (1938) *weak axiom of revealed preference* (WARP), which requires $f(T; B) = a$ if $f(T; A) = a$ and $a \in B \subseteq A$. Since WARP characterizes rational choice in our setting, it is incompatible with the requirement that f satisfies Faithfulness.

3 Two-stage majoritarian rules

We propose a class of choice rules that satisfy Faithfulness, Expansion Consistency and Weak WARP. Like the rules from Examples 1 and 2, each of our rules relies on an exogenous priority $P \in \mathcal{P}$. For our rules, the role of the linear ordering P is to partition the given tournament $T \in \mathcal{T}$ into two acyclic binary relations $T \cap P$ and $T \setminus P$. The first of these relations is used to obtain a preliminary shortlist of the feasible alternatives in the agenda $A \in \mathcal{X}$ while the second is used to make a final choice from the shortlist. Formally, the *two-stage majoritarian choice rule* f_P *based on the priority* $P \in \mathcal{P}$ is defined, for all $T \in \mathcal{T}$ and $A \in \mathcal{X}$, by

$$f_P(T; A) := \max(T \setminus P; \max(T \cap P; A)). \tag{1}$$

[8]The same choice patterns arise under the *amendment procedure* a_P (Miller, 1977, p. 779; Moulin, 1986, p. 287). Following our convention (that higher-ranked alternatives in P are more privileged), the linear ordering $P = 1, 2, 3, 4$ corresponds to the tree $\Gamma_4(4, 3, 2, 1)$ in Moulin. For the tournament T given in Example 2, the corresponding choice function gives $a_P(T; A) = s_P(T; A)$ for all $A \in \mathcal{X}$.

For each tournament $T \in \mathcal{T}$, the choice function $f_P(T; \cdot)$ is a *rational shortlist method* in the sense of Manzini and Mariotti (2007). Recall that a choice function $c : \mathcal{X} \to X$ is a rational shortlist method if there is a pair of asymmetric binary relations (P_1, P_2) (called *rationales*) on X such that $c(A) = \max(P_2; \max(P_1; A))$ for all $A \in \mathcal{X}$. To ensure that c is a well-defined choice function, the rationales P_1, P_2 must satisfy nontrivial restrictions (see Lemma 2 of Dutta and Horan, 2015). By way of contrast, $f_P(T; \cdot)$ is a well-defined choice function for *all* possible $T \in \mathcal{T}$ and $P \in \mathcal{P}$.

To see this, fix an agenda $A \in \mathcal{X}$. Since the binary relation $T \cap P$ is acyclic, the shortlist $M_A := \max(T \cap P; A)$ must be nonempty. The alternatives excluded from M_A are those dominated *both* in terms of the majority tournament T and the priority ordering P. That is, the shortlist M_A consists of those alternatives that are not majority beaten by any higher priority alternatives. It follows that the restriction of the relation $T \setminus P$ *to* M_A must be a linear ordering. To see this, define the linear ordering $P^{-1} := \{(a, b) \in X^2 \mid (b, a) \in P\}$ and observe that for all $a, b \in M_A$,

$$aTb \Leftrightarrow a(T \setminus P)b \Leftrightarrow a(T \cap P^{-1}) \Leftrightarrow aP^{-1}b.$$

This chain of equivalences shows that formula (1) can be re-written as

$$f_P(T; A) = \max(T; M_A) \quad \text{or even} \quad f_P(T; A) = \max(P^{-1}; M_A).$$

In other words, the alternative selected from the shortlist is the alternative most preferred by the majority. Equivalently, it is also the shortlisted alternative with lowest priority.

Finally, note that if the tournament T disagrees with the priority P for all pairs of alternatives in X (i.e., $T \cap P = \emptyset$), then the shortlist M_A is just A itself. Since $T = P^{-1}$ is a linear ordering in that case, it follows that $f_P(T; A)$ is the Condorcet winner of T in A. At the other extreme where the tournament T and the priority P coincide (i.e., $T = P$), the shortlist M_A contains only the Condorcet winner of T in A, which must then be selected in the second stage.

The following example serves as further illustration of the rules that we propose.

Example 3 (Two-stage majoritarian rules). *For the tournament T from Example 2, the two rationales associated with the priority $P = 1, ..., 4$ are*

$$P_1 = T \cap P = \{(1, 3), (1, 4), (2, 4)\} \quad \text{and} \quad P_2 = T \setminus P = \{(2, 1), (3, 2), (4, 3)\}.$$

To understand the resulting two-stage majoritarian rule f_P, first consider the Condorcet cycle $A = \{1, 2, 3\}$. Since $1P_13$, the first stage eliminates alternative 3, which gives the shortlist $\{1, 2\}$. Since $2P_21$, the second stage eliminates alternative 1, so that $f_P(T; A) = 2$ is the final choice.

Letting $f_P^{-1}(T;x) := \{A \in \mathcal{X} \mid f(T;A) = x\}$, the same kind of reasoning establishes that:

$$\begin{aligned}
f_P^{-1}(T;1) &= \{\{1,3\},\{1,4\}\}\,, \\
f_P^{-1}(T;2) &= \{\{1,2\},\{2,4\},\{1,2,3\},\{1,2,4\},\{1,2,3,4\}\}\,, \\
f_P^{-1}(T;3) &= \{\{2,3\},\{2,3,4\}\}\,, \text{ and} \\
f_P^{-1}(T;4) &= \{\{3,4\}\}\,.
\end{aligned}$$

By definition, every two-stage majoritarian rule f_P satisfies Faithfulness. Since the choice function $f_P(T;\cdot)$ is a rational shortlist method for each $T \in \mathcal{T}$, Manzini and Mariotti's characterization implies that f_P must also satisfy Expansion Consistency and Weak WARP.[9]

Of course, the rule f_P also satisfies properties beyond these. For a *fixed* tournament $T \in \mathcal{T}$, $f_P(T;\cdot)$ defines a rational shortlist method with *acyclic* rationales. Houy (2008) shows that this feature limits the scope of choice behavior. He also shows that, in order to characterize the strict sub-class of rational shortlist methods with acyclic rationales, it is necessary and sufficient to strengthen Manzini and Mariotti's Weak WARP to an acyclicity condition reminiscent of the *strong axiom of revealed preference* (SARP). Naturally, f_P must also satisfy this property.

The rule f_P also possesses consistency properties *across* tournaments that cannot be not captured in the standard choice setting that Houy considers. These properties follow from the fact that the *same* linear ordering P is used to partition each tournament T into two acyclic relations. One such property is an adaptation of Arrow's IIA to our setting (due to Moulin, 1986, p. 278). Let $T|_A$ denote the restriction of the tournament $T \in \mathcal{T}$ to the agenda $A \in \mathcal{X}$.

Choice IIA. *For all $T, T' \in \mathcal{T}$ and $A \in \mathcal{X}$ such that $T|_A = T'|_A$: $f(T;A) = f(T';A)$.*

To paraphrase, the majority view of infeasible alternatives cannot affect choice. Not only is this property satisfied by two-stage majoritarian rules, it is also satisfied by the rules from Examples 1 and 2 (as well as the variations of these rules discussed in footnotes 6 and 8).

Another inter-tournament consistency property satisfied by f_P, which is not satisfied by any of the other rules discussed in Section 2, is that choice must improve when all majority comparisons are reversed. Where $T^{-1} := \{(a,b) \in X^2 \mid (b,a) \in T\}$ denotes the reversal of tournament T,

Reversal Improvement. *For all $T \in \mathcal{T}$ and $A \in \mathcal{X}$: $f(T;A)\,T\,f(T^{-1};A)$.*

This property strengthens Faithfulness, which coincides with the special case where $|A| = 2$. It also strengthens a condition that Saari and Barney (2003, p. 17) proposed for the richer setting where collective choice may depend on individual preferences. Their condition requires the collective

[9] Rubinstein and Salant (2008) characterize rational shortlist methods with a different property called *Exclusion Consistency*, which can be adapted to our setting as follows: *for all $T \in \mathcal{T}$, $a \in X$, and $A, B \in \mathcal{X}$ such that $a \in B \setminus A$, $f(T;A\cup\{a\}) \notin \{f(T;A), a\}$ implies $f(T;B) \neq f(T;A)$.* In all of our subsequent analysis, this property can be used in place of Expansion Consistency and Weak WARP.

choice to change when all individual preferences are reversed.[10] In our setting, this amounts to the requirement that $f(T;A) \neq f(T^{-1};A)$.

Reversal Improvement requires that reversing preferences must not only change choice but must *improve* it. What motivates us to strengthen Saari and Barney's condition in this way is the view that changes to the majority view should impact choice for the better. This makes Reversal Improvement similar, at least conceptually, to May's positive responsiveness (which we discuss at greater length in Section 4.2 below). The main difference is that May's condition relates to changes that reinforce the support for a particular choice. In contrast, our condition relates to changes that reverse all comparisons that led to a particular choice.

In combination with Expansion Consistency and Weak WARP, Choice IIA and Reversal Improvement characterize two-stage majoritarian rules. To state our result formally:

Theorem. *A choice rule $f : \mathcal{T} \times \mathcal{X} \to X$ is a two-stage majoritarian choice rule if and only if it satisfies Expansion Consistency, Weak WARP, Choice IIA and Reversal Improvement.*

Proof. Necessity of Expansion Consistency and Weak WARP follow from Manzini and Mariotti (2007). Choice IIA is also immediate. To see that f_P satisfies Reversal Improvement, fix some $T \in \mathcal{T}$ and $A \in \mathcal{X}$. If $f_P(T^{-1};A)Pf_P(T;A)$, then $f_P(T;A)Tf_P(T^{-1};A)$ since $f_P(T;A) \in \max(T \cap P, A)$. Similarly, if $f_P(T;A)Pf_P(T^{-1};A)$, then $f_P(T^{-1};A)T^{-1}f_P(T;A)$. Finally, if $f_P(T;A) = f_P(T^{-1};A) = a$, then $a \in \max(T \cap P;A) \cap \max(T^{-1} \cap P;A)$. It follows that aPc for all $c \in A \setminus \{a\}$. Let b be the second-ranked alternative in A according to P. Since $f_P(T;A) = f_P(T^{-1};A) = a$, $b \notin \max(T \cap P;A) \cup \max(T^{-1} \cap P;A)$. So, aTb and $aT^{-1}b$, which is a contradiction.

Since sufficiency is immediate if $|X| = 2$, suppose $|X| \geq 3$. Define the binary relation R on X such that, for all $x, y \in X$: xRy if there is some $T \in \mathcal{T}$ and $z \in X$ such that $xTzTyTx$ and $f(T;\{x,y,z\}) = x$. Equivalently, by Reversal Improvement, xRy if there is some $T' \in \mathcal{T}$ and $z \in X$ such that $xT'yT'zT'x$ and $f(T';\{x,y,z\}) = z$. We write xIy if neither xRy nor yRx.

Step 1. *R is (i) asymmetric and (ii) transitive.*

(i) To the contrary, suppose $xRyRx$ for some $x, y \in X$. By definition, there are $c, d \in X \setminus \{x,y\}$ and $T, T' \in \mathcal{T}$ such that $xTyTcTx$, $xT'yT'dT'x$, $f(T;\{x,y,c\}) = c$, and $f(T';\{x,y,d\}) = y$. By Choice IIA, it must be that $c \neq d$. For $|X| = 3$, this yields a contradiction directly. For $|X| \geq 4$, consider $T^* \in \mathcal{T}$ such that $T^*|_C = T|_C$ for $C := \{x,y,c\}$, $T^*|_D = T'|_D$ for $D := \{x,y,d\}$, and cT^*d. By Faithfulness, $f(T^*;\{c,d\}) = c$ and $f(T^*;\{y,d\}) = y$. Since $f(T^*;C) = c$ and $f(T^*;D) = y$ by Choice IIA, Expansion Consistency leads to the following contradiction:

$$c = f(T^*; C \cup \{c,d\}) = f(T^*;\{x,y,c,d\}) = f(T^*; D \cup \{y,d\}) = y.$$

[10] Fishburn (1973, p. 157) earlier proposed a similar condition, which he called *Duality*.

(ii) Suppose $xRyRz$. Consider $T \in \mathcal{T}$ such that $xTyTzTx$. If $f(T;\{x,y,z\}) \neq x$, yRx or zRy. Since $xRyRz$, this contradicts the asymmetry of R. So, $f(T;\{x,y,z\}) = x$ and xRz. $\square$

Step 2. *There are exactly two distinct $a, b \in X$ such that aIb; and a, bRc for all $c \in X \setminus \{a,b\}$.*

Suppose there are pairs of distinct alternatives $\{x,y\}$ and $\{z,w\}$ (with $x \neq z, w$ and $z \neq x, y$) such that xIy and zIw. First, consider $T \in \mathcal{T}$ such that $xTzTwTx$ and its reversal T^{-1}. Since zIw, the definition of R implies zRx and wRx. Next, consider $T \in \mathcal{T}$ such that $xTyTzTx$ and its reversal T^{-1}. Then, xRz since xIy. Since $xRzRx$ contradicts the asymmetry of R, it follows that aIb for at most one pair $\{a,b\}$.

If there is no such pair, then R is a linear ordering by Step 1. Suppose aRb and bRc for all $c \in X \setminus \{a,b\}$. Then, by definition of R, there is some $d \in X \setminus \{a,b\}$ such that dRb. Since this is a contradiction, there must be exactly one pair $\{a,b\}$ such that aIb. Finally, from the definition of R, it follows that aRc and bRc for all $c \in X \setminus \{a,b\}$. $\square$

Complete R into a linear ordering P by defining aPb and xPy if xRy for $x, y \in X$.

Step 3. *For all $T \in \mathcal{T}$ and $A \in \mathcal{X}$, $f(T;A) = f_P(T;A)$.*

The proof is by strong induction on $|A|$. For the base cases $|A| = 2, 3$, the result follows from Reversal Improvement, Expansion Consistency, and the definition of R. For the induction step, suppose that the result holds for $2 \leq |A| < n$ and consider the case $|A| = n \geq 4$.

By the induction hypothesis, it suffices to show that $f(T;A) = f_P(T;A)$ for all $T \in \mathcal{T}$. Labelling the alternatives of $A = \{a_1, ..., a_n\}$ so that $a_1P...Pa_n$, this is equivalent to showing that

$$f(T;A) = \begin{cases} a_n & \text{if } a_n = \max(T;A), \\ f(T;A \setminus \{a_n\}) & \text{otherwise.} \end{cases} \tag{2}$$

First, suppose $f(T;A \setminus \{a_n\}) = x$ and recursively define a sequence $\langle b_i \rangle_{i=0}^m$ in A such that:

(i) $b_0 := a_n$;
(ii) $B_{i+1} := \{y \in A \mid y(P \cap T)b_i\}$ and $b_{i+1} := \max(P; B_{i+1})$; and,
(iii) m is the smallest index such that b_mPx, $b_m = x$, or $B_{m+1} = \emptyset$.

Next, define $B := \{b_0, ..., b_m\}$. Since $f(T;A \setminus \{a_n\}) = x$, there are two possibilities. If a_nTa_i for all $i = 1, ..., n-1$, then $B = \{a_n\} = \{b_0\}$. Otherwise, $B = \{b_0, ..., b_m\} \neq \{b_0\}$ with the features that: (a) $b_m(T \cap P)...(T \cap P)b_0$; and (b) $x(T \setminus P)b_m$ or $x = b_m$. This leaves three cases:

Case 1. If $B = \{a_n\}$, then $a_n = \max(T;A)$. By the induction hypothesis and the application of Expansion Consistency to $f(T;\{a_i, a_n\}) = a_n$ for $i = 1, ..., n-1$, it follows that $f(T;A) = a_n$.

Case 2. If $1 < |B \cup \{x\}| < n$, then $f(T;B \cup \{x\}) = x$ by (a)-(b) above and the induction hypothesis. So, $f(T;A) = f(T;(B \cup \{x\}) \cup (A \setminus \{a_n\})) = x$ by Expansion Consistency.

Case 3. If $B \cup \{x\} = A$, then $x \in \{a_1, a_2\}$ by (a)-(b) above. What is more, the definition of B implies: (c) $a_{i-1}Ta_i$ for $i = 4, ..., n$; and (d) a_iTa_j for $i = 4, ..., n$ and all $j < i - 1$.

First, suppose $x = a_1$. By definition of B, $a_1Ta_2Ta_3Ta_1$. Given (c)-(d), $a_{i-2}Ta_{i-1}Ta_iTa_{i-2}$ for $i = 3, ..., n$. So, $f(T; \{a_{i-2}, a_{i-1}, a_i\}) = a_{i-2}$ by the induction hypothesis. Since $f(T; \{a_{i-2}, a_i\}) = a_i$ by the induction hypothesis, Weak WARP precludes $f(T; A) = a_i$. So, $f(T; A) \in \{a_1, a_2\}$.

To rule out $f(T; A) = a_2$, consider the reversal T^{-1} of T. Since $a_{i-2}T^{-1}a_iT^{-1}a_{i-1}T^{-1}a_{i-2}$ for $i = 3, ..., n$, the same kind of argument given for T implies $f(T^{-1}; A) \in \{a_1, a_2\}$. Since $f(T; A) \in \{a_1, a_2\}$ and a_1Ta_2, Reversal Improvement then implies $f(T; A) = a_1 = x$.

Next, suppose $x = a_2$. From the definition of B and the fact that $f(T; (A \setminus \{a_n\}) = a_2$, it follows that a_1Ta_3 and a_2Ta_1. We distinguish two possibilities: (i) a_3Ta_2; and (ii) a_2Ta_3.

(i) Given $a_2Ta_1Ta_3Ta_2$ and (c), the same kind of argument used for $x = a_1$ establishes $f(T; A) = a_2$. (The difference is that $f(T; A) = a_3$ is ruled out by $a_2Ta_1Ta_3Ta_2$ while $f(T; A) = a_4$ is ruled out by $a_1Ta_3Ta_4Ta_1$. In turn, $f(T^{-1}; A) = a_3$ is ruled out by $a_2T^{-1}a_3T^{-1}a_1T^{-1}a_2$ while $f(T^{-1}; A) = a_4$ is ruled out by $a_1T^{-1}a_4T^{-1}a_3T^{-1}a_1$.)

(ii) By the induction hypothesis: $f(T; A \setminus \{a_1\}) = a_2$ given a_2Ta_3 and (c); and $f(T; \{a_1, a_2\}) = a_2$ given a_2Ta_1. So, $f(T; A) = f(T; (A \setminus \{a_1\}) \cup \{a_1, a_2\}) = a_2 = x$ by Expansion Consistency. ■

Remark. The proof shows that exactly <u>two</u> different priorities (denoted P_a and P_b below) may be used to represent a choice rule satisfying the axioms. These priorities differ only in terms of how they rank the top two alternatives a and b from Step 2 of the proof. To see why this non-uniqueness is inherent in the model, note that formula (2) implies $f_{P_a}(T; \{a, b\}) = \max(T; \{a, b\}) = f_{P_b}(T; \{a, b\})$ for all $T \in \mathcal{T}$. As a result, $f_{P_a}(T; A) = f_{P_b}(T; A)$ for all $T \in \mathcal{T}$ and $A \in \mathcal{X}$.

4 Further remarks

4.1 Flexibility and Pareto sub-optimality

The well-known *top cycle* correspondence $TC : \mathcal{T} \times \mathcal{X} \to \mathcal{X}$ (Camion, 1959; Good, 1971; Schwartz, 1972; Smith, 1973; Fishburn, 1974) is defined, for all $T \in \mathcal{T}$ and $A \in \mathcal{X}$, by $TC(T; A) := \{a \in A \mid \forall b \in A \setminus \{a\} : a = c_1T...Tc_n = b \text{ for some } c_1, ..., c_n \in A\}$. Just like the uncovered set, the top cycle correspondence is Condorcet-consistent.

For all $T \in \mathcal{T}$ and $A \in \mathcal{X}$, the set of alternatives chosen by a two-stage majoritarian rule coincides with the top cycle, that is $TC(T; A) = \{f_P(T; A) \mid P \in \mathcal{P}\}$. To see that $TC(T; A) \supseteq \{f_P(T; A) \mid P \in \mathcal{P}\}$, pick any $P \in \mathcal{P}$ and note that $f_P(T; A) = f_P(T; TC(T; A)) \in TC(T; A)$. For the converse inclusion, pick any $a \in TC(T; A)$. Then, it is possible to choose a path $a = a_1T...Ta_m$ such that $\{a_1, ..., a_m\} = A$ (see e.g., Lemma 8.3.3 of Laslier, 1997). Fix a priority $P \in \mathcal{P}$ such that $a_1P...Pa_m$. Since $\max(T \cap P; A) = a$ by construction, it follows that $f_P(T; A) = a$.

A classic result of Miller (1977) establishes that the same is true for successive elimination rules (Example 2), that is $TC(T; A) = \{s_P(T; A) \mid P \in \mathcal{P}\}$. This means that two-stage majoritarian rules provide the same *flexibility* to the designer as successive elimination rules.

It is well known that, for some tournaments $T \in \mathcal{T}$ and agendas $A \in \mathcal{X}$ such that $|A| \geq 4$, the top cycle $TC(T; A)$ contains alternatives that are Pareto dominated at some preference profiles consistent with T. This implies that all two-stage majoritarian rules occasionally make Pareto sub-optimal choices. To illustrate, suppose $X = \{1, 2, 3, 4\}$ and consider the two-stage majoritarian rule f_P based on the priority $P = 4, 3, 2, 1$. Suppose (as in Bordes, 1979, p. 188) that there are three agents with preferences $\succ_1 = 1, 4, 3, 2$, $\succ_2 = 2, 1, 4, 3$, and $\succ_3 = 3, 2, 1, 4$. The majority tournament T generated by this profile is the one from Example 2. While alternative 4 is Pareto-dominated by alternative 1, it is not difficult to see that $f_P(T; X) = 4$.

4.2 The connection to May

Two-stage majoritarian rules satisfy a natural adaptation of May's *positive responsiveness* to the tournament setting. To state this adaptation (originally formulated by Moulin, 1986, p. 285), say that a binary relation $R^{\uparrow a}$ on X *improves* an alternative $a \in X$ relative to another binary relation R on X if, for all $x, y \in X \setminus \{a\}$: (i) $aRx \Rightarrow aR^{\uparrow a}x$; and (ii) $xRy \Leftrightarrow xR^{\uparrow a}y$.

T-Monotonicity. *For all $T \in \mathcal{T}$, $a \in X$, $T^{\uparrow a} \in \mathcal{T}$ that improves a, and $A \in \mathcal{X}$:*

$$f(T; A) = a \textit{ implies } f(T^{\uparrow a}; A) = a.$$

In other words: improving the majority view of a chosen alternative must reinforce its choice.

To see that two-stage majoritarian rules satisfy this property, recall that $f_P(T; A)$ is the lowest priority alternative in A that beats all higher priority alternatives by majority. Improving $f_P(T; A)$ relative to T does not change this: $f_P(T; A)$ still beats all higher priority alternatives; and every alternative with lower priority is still beaten by some higher priority alternative.

It is known that the rules from Example 2 (just like the rules from footnotes 6 and 8) also satisfy T-Monotonicity.[11] However, the rules from Example 1 do not. To illustrate, consider $X = \{1, 2, 3, 4\}$ and the priority $P = 4, 3, 2, 1$. Then, $UC_P(T; X) = 3$ for the tournament T from Example 2 while $UC_P(T'; X) = 4$ for the tournament T' that improves 3 relative to 1.

[11] See Exercise 9.4(c) of Moulin (1988, p. 250) for s_P and the Corollary to Theorem 9.5 (p. 247) for a_P. Horan (2021) shows that a much broader range of binary trees (which he calls "simple agendas") have the same feature. For TC_P, simply note that the top cycle cannot gain new members by improving one of its members.

4.3 The role of the priority

A minor variation on the argument used to show that two-stage majoritarian rules satisfy T-Monotonicity also establishes that every rule f_P is monotonic with respect to the priority P. In other words, two-stage majoritarian rules satisfy the following property:

P-**Monotonicity.** *For all $P \in \mathcal{P}$, $a \in X$, $P^{\uparrow a} \in \mathcal{P}$ that improves a, $T \in \mathcal{T}$, and $A \in \mathcal{X}$:*

$$f_P(T;A) = a \textit{ implies } f_{P^{\uparrow a}}(T;A) = a.$$

This property captures the idea that alternatives ranked higher by P are more privileged.

The rules from Examples 1 and 2 (as well as the related rules from footnotes 6 and 8) satisfy an analogous property.[12] The difference is that the priority P plays a less intrusive role for two-stage majoritarian rules than for these other rules. To see this, first consider the successive elimination rules from Example 2. By definition, the chosen alternative $s_P(T;A)$ must defeat all higher priority alternatives in the agenda $A \in \mathcal{X}$. Because the same is true for $f_P(T;A)$, this means that, when the alternatives $s_P(T;A)$ and $f_P(T;A)$ differ, $f_P(T;A)$ must be preferred by a majority over $s_P(T;A)$. The same reasoning shows that $f_P(T;A)$ must be weakly preferred by majority to the alternatives $TC_P(T;A)$ and $a_P(T;A)$ chosen by the top cycle selection rule (footnote 6) and the amendment rule (footnote 8). Indeed, the same is true for the uncovered set selection rule from Example 1 once differences in flexibility of the two rules are taken into account: if $f_P(T;A) \in UC(T;A)$, then $f_P(T;A)$ must be weakly preferred by a majority to $UC_P(T;A)$.

Nonetheless, the choice of the priority P has a significant impact on the outcomes associated with the rule f_P. Fortunately, there is a natural way to define the priority in many collective choice settings. In the public policy setting, for instance, it is conventional to prioritize policies that are less costly or, in some cases, more equitable. In legislative settings, it may be more natural to follow the convention of prioritizing proposals either by the order in which they were tabled or by their degree of divergence from the *status quo* legislation (Rasch, 2000, p. 15). Finally, in the committee setting, it may be reasonable to use the preference of the chair, which is conventionally used as a tie-breaking device (Robert, 2011, p. 405), to define the priority.

4.4 An extension

There may be settings where it is desirable to select the same alternative both for a tournament and its reversal. To accommodate this possibility, it is necessary to weaken the conclusion of Reversal Improvement to allow $f(T;A) = f(T^{-1};A)$. Unlike Reversal Improvement, the resulting *Weak Reversal Improvement* property does not imply Faithfulness.

[12] For UC_P and TC_P, the claim is straightforward. For s_P and a_P, see Exercise 9.5 of Moulin (1988, p. 250). A much broader class of binary trees introduced by Horan (2021) (called "priority agendas") have the same feature.

When combined with Faithfulness and the other requirements in our Theorem, Weak Reversal Improvement defines a much broader class of choice rules. The next example describes some rules in this class that share the same basic structure as two-stage majoritarian rules.

Example 4. (General two-stage majoritarian rules) *Let* $\mathcal{R}_2$ *denote the set of weak orderings*[13] R *on* X *such that, for any* $x \in X$*, the indifference class* $I_R(x) := \{y \in X \mid xRyRx\}$ *contains at most two alternatives. Given a weak ordering* $R \in \mathcal{R}_2$*, let* g_R *be the choice rule defined, for all* $T \in \mathcal{T}$ *and* $A \in \mathcal{X}$*, by*

$$g_R(T; A) := \max(T \setminus R; \max(T \cap R; A)). \tag{3}$$

To see that g_R *does indeed define a choice rule, let* R_T *denote the linear ordering obtained by taking the lexicographic composition of* R *with a tournament* $T \in \mathcal{T}$*. Then,* $g_R(T; A) = f_{R_T}(T; A)$ *for all* $A \in \mathcal{X}$*. Not only does this show that* g_R *is well-defined, it shows that* g_R *is a two-stage majoritarian rule when* R *contains no indifferences (since, in that case,* $R_T = R_{T'}$ *for all* $T, T' \in \mathcal{T}$*). This is not true when* $xRyRx$ *for distinct* $x, y \in X$*. Then,* x *and* y *must be compared by the first rationale* $T \cap R$ *regardless of* $T \in \mathcal{T}$*, something which cannot occur for a two-stage majoritarian rule.*

Besides Faithfulness, Expansion Consistency, Weak WARP, Choice IIA, and Weak Reversal Improvement, g_R satisfies T-Monotonicity and (the analog of) P-Monotonicity for any weak ordering $R \in \mathcal{R}_2$. To see that g_R may violate Reversal Improvement when $|X| \geq 3$, consider a weak ordering $R \in \mathcal{R}_2$ such that $1R2R3R2$ and a tournament $T \in \mathcal{T}$ such that $1T2T3T1$. Then, by definition of g_R, it follows that $g_R(T; A) = 1 = g_R(T^{-1}; A)$ for the agenda $A = \{1, 2, 3\}$.

To close, we note that the rules from Example 4 provide the same flexibility as two-stage majoritarian rules, that is $TC(T; A) = \{g_{\sigma R}(T; A) \mid \sigma \text{ is a permutation on } X\}$ for all $R \in \mathcal{R}_2$, $T \in \mathcal{T}$, and $A \in \mathcal{X}$. As in the case of two-stage majoritarian rules, the implication is that all of these rules occasionally make Pareto sub-optimal choices. This raises the question of whether an efficient choice rule is capable of satisfying all of the desiderata listed in the last paragraph.

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