Agendas in Legislative Decision-Making

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Abstract

Despite the wide range of agendas used in legislative decision-making, the literature has focused almost exclusively on two stylized formats — the so-called Euro-Latin and Anglo-American agendas. As emphasized by Ordeshook and Schwartz [1987], this focus leaves a sizable gap in our understanding of the legislative process.

To address this deficiency, I first define a broad class of agendas (called simple agendas) whose defining structural features are common among legislative agendas; and, then characterize the sophisticated voting (Farquharson [1969]) outcomes implemented by agendas in this class. By establishing a clear connection between the structure of simple agendas and the policies that they implement, the characterization extends our understanding of legislative decision-making well beyond the limited scope of Euro-Latin and Anglo-American agendas.

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1 Introduction

Agendas provide a straightforward way to make group decisions. By majority voting on the sequence of “aye” (yes) or “nay” (no) questions specified by an agenda, the group can whittle down the issue progressively and, ultimately, arrive at an outcome which enjoys broad support.

In legislative settings, almost every issue is decided by agenda. Due to their critical role in public decision-making, agendas have attracted significant interest from economists and political scientists. Starting with Black [1948, 1958], the goal has been to understand how the structure of the agenda influences the outcome. Subsequently, Farquharson [1969] focused the analysis on strategic voting behavior specifically. He identified two formats — sometimes called Euro-Latin and Anglo-American agendas — that can arise in legislative settings and sought to understand the sophisticated voting outcomes associated with each.\footnote{In the current paper, I follow Apesteguia, Ballester, and Masatlioglu [2014]. In Farquharson’s work (and elsewhere in the literature), Euro-Latin agendas are called successive agendas and Anglo-American agendas are called amendment agendas.}

\begin{align*}
\text{Q1.} & \quad \text{Shall } x_1 \text{ be accepted?} \\
\text{Q2.} & \quad \text{If not, shall } x_2 \text{ be accepted?}
\end{align*}

\begin{align*}
\text{Q1.} & \quad \text{Shall } x_1 \text{ be selected over } x_2? \\
\text{Q2.} & \quad \text{Shall the “winner” of Q1. be selected over } x_3?
\end{align*}

\begin{figure}[h]
\centering
\begin{tikzpicture}
\t为客户编码
\end{tikzpicture}
\caption{Left: a Euro-Latin agenda on three alternatives. Right: an Anglo-American agenda on three alternatives.}
\end{figure}

While Farquharson’s analysis did not go beyond three alternatives, subsequent work has provided us with a clear understanding of sophisticated voting on Euro-Latin and Anglo-American agendas (with three or more alternatives). In early contributions, Miller [1977, 1980] and Banks [1985] characterized the range of outcomes associated with each of these formats. More recently, Apesteguia, Ballester, and Masatlioglu [2014] axiomatized the rules implemented by Euro-Latin and Anglo-American agendas — showing how the outcome associated with each format varies with the voters’ preferences and the set of feasible alternatives.

Initially, the focus on Euro-Latin and Anglo-American formats was motivated by the fact that they “approximate” the standard legislative procedure in these two legal traditions (see Miller [1977, p.780]). This begs the empirical question: how “closely” do they approximate the agendas used in practice? Illustrating with a variety examples, Ordeshook and Schwartz [1987] argue that legislative agendas regularly depart from the Euro-Latin and Anglo-American formats. (Figures 3-4 below present a few of their examples while some others are discussed in Section 3.1.) While a systematic study of legislative practices has never been undertaken, the
qualitative evidence — drawn from surveys (Sullivan [1984]; Miller [1995]; Schwartz [2008]) and case studies (Bach [1983]; Krehbiel and Rivers [1990]; Calvert and Fenno [1994]) — tends to support their view. The upshot is that agenda voting can lead to outcomes that are quite different from those associated with Euro-Latin and Anglo-American agendas. This is troubling. Agenda voting plays a critical role in legislative decision-making; and, without a clear understanding of this institution, one cannot begin to address some of the most basic questions in political economy or public economics.

With this in mind, the goal of the current paper is to extend our understanding of sophisticated voting to a much wider range of legislative agendas beyond Euro-Latin and Anglo-America agendas. To do so, I first define a broad class of agendas, called simple agendas, whose two defining features are prevalent among agendas used in practice. I also define a special sub-class, called priority agendas, whose sequential structure is most closely related to Euro-Latin and Anglo-America agendas.

To get a better idea of these agenda classes, recall that an agenda is a sequence of questions to be decided by (simple) majority voting. Formally, the sequence defines a binary game tree where each stage game amounts to a vote between “aye” and “nay” options. Figure 1 above illustrates the questions (top) and game trees (bottom) associated with the Euro-Latin and Anglo-American agendas on three alternatives. For a Euro-Latin agenda (left), the voting is by sequential majority approval: in each stage, the voters consider a single alternative. Ultimately, the alternative selected is the first one to be approved by majority. For an Anglo-American agenda (right), the voting is by sequential majority comparison: in each stage, the voters compare two alternatives with the “winner” moving on to the next comparison and the “loser” being eliminated. Ultimately, the alternative selected is the only one which survives this process.

Despite their differences, Euro-Latin and Anglo-American agendas share the same kind of structure. In particular, both can be built up from a priority ordering, which specifies when to add each alternative to the agenda, and an amendment rule, which specifies how to add each alternative to the agenda.

The agendas in Figure 1 have the same priority ordering: $x_1$ is the first alternative added to the agenda, $x_2$ is the second, and $x_3$ is the third. The difference between the two is the amendment rule: for the Euro-Latin agenda, the last alternative $x_3$ can only ever face the alternative $x_2$; and, for the Anglo-American agenda, this alternative can face either $x_1$ or $x_2$. Both of these amendment rules extend naturally to longer agendas: for a Euro-Latin agenda, each new addition can only face the last alternative added to the agenda; and, for an Anglo-American agenda, it can face each of the alternatives previously added to the agenda. Figure 2 illustrates how to extend the two agendas in Figure 1 when a fourth alternative ($x_4$) is added.

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2The only real outlier is Rasch [2000], which concludes that Euro-Latin and Anglo-American agendas are the dominant formats. However, his study is “not based on a close inspection of voting practices” and does not show “how stringently the two procedures are applied in day-to day proceedings” (see p.8). For further discussion of Rasch’s methodology, see Schwartz [2008, p.368].

3The priority does not necessarily capture the temporal order in which motions are proposed. As illustrated by Figure 3, the agenda may be backwards-built, a practice which is relatively common in the Anglo-American legal tradition (see Miller [1995]).

4Whether it involves a perfecting amendment, a second-order amendment, or a substitute affects how the alternative is added.

5To avoid clutter, I omit the edge labels. Instead, I adopt the convention that left means “aye” and right means “nay”.

For Euro-Latin and Anglo-American agendas, the same style of amendment occurs at each stage. One way to expand the set of agendas is to allow for a "mix" of Euro-Latin and Anglo-American style amendments. Figure 3 illustrates two such "hybrid agendas" (as they are called by Ordeshook and Schwartz).

Priority agendas generalize the mixed amendment structure of hybrid agendas. Like Euro-Latin and Anglo-American agendas, they are defined by a priority ordering and an amendment rule. The difference is that they allow for a much wider range of amendments: a new addition \( z \) can face any alternative \( y \) that is already on the agenda — provided that it can also face each alternative that was added between \( y \) and \( z \).

Because of their flexible structure, priority agendas include a wide variety of agendas used in practice. To illustrate, suppose that a bill \( b \) is put up for consideration and, subsequently, two amendments \( a' \) and \( a'' \) are proposed. If the two amendments are incompatible (in the sense that both of these proposed modifications cannot be applied simultaneously to \( b \)), then US Congressional procedure requires the following sequence of questions (for a more detailed exposition, see Figure 5 in Ordeshook and Schwartz [1987]):
Q1. Shall the first amendment \(a'\) be accepted?
Q2. If not, shall the second amendment \(a''\) be accepted?
Q3. Shall the bill as amended be selected over the status quo?

This sequence corresponds to the left-hand agenda in Figure 3 where: \(x_4\) represents the status quo (or current) legislation; \(x_3\) represents the bill \(b\); and, \(x_2\) (resp. \(x_1\)) represents the bill as amended by \(a''\) (resp. \(a'\)).

Intuitively, this agenda uses Euro-Latin procedure to determine which version of the bill \((x_1, x_2, \text{ or } x_3)\) is put to a final up or down vote. A different approach would be to first recognize only one of the two amendments as a “question of principle” before using Euro-Latin procedure to decide among the remaining legislative options. This approach is captured by the following sequence of questions:

Q1. Shall the amendment \(a'\) be recognized over the amendment \(a''\)?
Q2. Shall the bill as amended (by the “winner” of Q1.) be accepted?
Q3. If not, shall the original bill be accepted?

It is not difficult to see that this sequence of questions corresponds to the right-hand agenda in Figure 3 (when the alternatives \(x_1\) to \(x_4\) are given the same interpretation specified above).

Despite their flexibility, priority agendas still rule out a variety of agendas used in practice. To illustrate, consider how the left-hand agenda in Figure 3 must be modified when the amendments \(a'\) and \(a''\) are compatible. In that case, US Congressional procedure requires an “up or down” vote on \(a'\) before \(a''\) can be recognized (for a more detailed exposition, see the discussion related to Figure 3 in Ordeshook and Schwartz [1987]; Figure 1.2 in Sullivan [1984]; or, Figure 3 in Miller [1995]). This leads to the agenda depicted in Figure 4:

Q1. Shall the first amendment \(a'\) be accepted?
Q2. Shall the second amendment \(a''\) be accepted?
Q3. Shall the bill as amended be selected over the status quo?

\[
\begin{align*}
\{x_0, x_1, x_2, x_3, x_4\} & \quad \{x_0, x_1, x_4\} & \quad \{x_2, x_3, x_4\} \\
\{x_0, x_4\} & \quad \{x_1, x_4\} & \quad \{x_2, x_4\} & \quad \{x_3, x_4\} \\
x_0 & \quad x_1 & \quad x_2 & \quad x_3 & \quad x_4
\end{align*}
\]

Figure 4: A two-period Congressional agenda.

Compared with the left-hand agenda in Figure 3, the only difference (which stems from Q2.) is the addition of the bill \(b\) as amended by \(a'\) and \(a''\). In Figure 4, this “doubly amended” bill is represented by \(x_0\). Despite its
close relationship to the left-hand agenda in Figure 3, this “two-period” Congressional agenda is not a priority agenda. Intuitively, the problem is that the highly symmetric structure of the agenda cannot be reconciled with the existence of a priority ordering among the alternatives in \( \{ x_0, \ldots, x_3 \} \).

To accommodate this agenda (among many others), I move away from trying to define agendas in terms of the underlying institutional procedure; and, instead focus on the structural features that typically emerge from this process. The simple agendas that I consider are defined by two features: history-independence, which stipulates that the question posed (at a given stage) must only depend on the alternatives not yet eliminated from consideration; and, persistence, which requires that the “safe” alternatives (which do not risk elimination) at given stage must be tested as a group by one of the subsequent questions.\(^6\)

Both features are very common in practice. Among the legislative agendas discussed in the literature, the only ones that violate history-independence involve procedural motions where nothing is eliminated from consideration; and, the only ones that violate persistence involve substitute amendments. Re-stated in terms of the agenda taxonomy introduced by Ordeshook and Schwartz [1987], procedural questions make an agenda repetitive while substitutes make it discontinuous. In the result, simple agendas include a wide range of legislative agendas dealing with substantive issues — with the notable exception of discontinuous agendas like two-stage amendment agendas (see Example 10 of the Supplementary Appendix; the paper by Banks [1989]; or, Figure 2 of Ordeshook and Schwartz [1987]) which are sometimes used by US Congress.

1.1 Results

The definition of simple agendas and priority agendas is only the first step of the analysis.

The main contribution is to characterize what decision rules are implemented by agendas in these classes: as one varies the voters’ preferences or the set of feasible alternatives, how does the sophisticated voting outcome change? Broadly, the point is to determine how the structure of an agenda relates to its voting outcomes.

A secondary contribution (made possible by the main contribution) is to compare the sophisticated outcomes for simple agendas and priority agendas to some other agenda classes — specifically, the broader class of non-repetitive and continuous agendas; and, the narrower classes of Euro-Latin and Anglo-American agendas. The point is to establish how the scope of outcomes expands as the range of permissible agendas is enlarged.

Before turning to the related literature, I discuss these two contributions in somewhat greater detail; and, briefly touch on their implications for some applications in political economy and public economics.

**Implementation:** In Section 3, I first show that two conditions, Issue Splitting and the Independence of Losing Alternatives, characterize sophisticated voting on simple agendas (Theorem 1). The first condition weakens Path Independence (Plott [1973]): it stipulates that the outcome can always be determined by splitting the decision problem into a pair of sub-issues.\(^7\) In turn, the second condition weakens the Independence of Irrelevant Alternatives\(^8\) (Chernoff [1954]; Radner and Marschak [1954]): it stipulates that the removal of an alternative

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\(^6\)The agenda in Figure 4 is clearly history-independent (since every node has a unique label). Likewise, it is persistent. To see this, note that \( x_4 \) is the only alternative which is ever safe; but, whenever that is the case, it is tested by a subsequent question.

\(^7\)It also weakens an axiom called Division Consistency (Apesteguia et al. [2014]) or Weak Separability (Xu and Zhou [2007]).

\(^8\)This “elimination-based” IIA condition was first proposed in the context of individual choice; and, it is formally distinct from...
with no majority appeal (called the *Condorcet loser*) cannot affect the outcome.\(^9\)

In Section 5, I then show that the narrower class of priority agendas satisfies an additional *Marginalization* property known to hold for Euro-Latin and Anglo-American agendas (Proposition 3): there is some alternative which is chosen only if it has majority support over every other alternative (Miller [1977]; Moulin [1991]). (As with Euro-Latin and Anglo-American agendas, the *marginal alternative* is the last one added to the agenda.) Combined with *Issue Splitting* and the *Independence of Losing Alternatives*, this *Marginalization* property characterizes sophisticated voting on priority agendas (Theorem 2).

**Comparison:** In Section 4, I first show that simple agendas satisfy two properties that distinguish them from other non-repetitive and continuous agendas (Propositions 1-2): *Preference Monotonicity*, which states that the outcome is unchanged when it improves in voters’ preferences; and, *Self-Consistency*, which states that the removal of an alternative yields an agenda that is outcome-equivalent to a simple agenda. Both properties limit the scope for strategic behavior on simple agendas. For the first property, the connection to incentives is clear — since it is similar to (but weaker than) *Strategyproofness* (Sanver and Zwicker [2009]) and May’s [1952] *Positive Responsiveness* (Fishburn [1982]). In turn, the second property restricts the ability to manipulate the structure of an agenda by adding “dummy” alternatives (with no majority appeal).

In Section 6, I then show that every property of Euro-Latin and Anglo-American agendas emphasized in the literature extends to all priority agendas. Besides Preference Monotonicity (Proposition 1), Self-Consistency (Proposition 2), and Marginalization (Proposition 3), priority agendas leave the outcome unchanged when it moves up in terms of priority (Proposition 4). Previously, this *Priority Monotonicity* property was only known to hold for Euro-Latin and Anglo-American agendas (Moulin [1986, 1991]; Jung [1990]). Nonetheless, there is a natural way to distinguish Euro-Latin and Anglo-American agendas from other priority agendas. It extends the idea that Euro-Latin agendas are less *discriminating* than Anglo-American agendas: the former can select any alternative in the Top Cycle (Miller [1977]) while the latter only selects from a special subset (identified by Banks [1985]). By generalizing the construction of Shepsle and Weingast [1984], I show that the discriminability of every other priority agenda must be somewhere between these two extremes (Proposition 5).

**Applications:** As emphasized by Schwartz [2008], the point of studying agenda voting is to provide insights that can be leveraged in applications that “embed” this institution in a broader strategic setting. In Section 7, I briefly highlight some potential implications of the results for applications ranging from strategic candidacy (Dutta et al. [2001, 2002]) to strategic agenda-setting (Dutta et al. [2004]; Duggan [2006]).

1.2 Related Literature

At one extreme, most of the literature on sophisticated voting contemplates a very narrow range of agendas. As discussed above, many of these papers (including the recent work of Gershkov, Moldovanu, and Shi [2015]; the IIA condition that Arrow [1950] proposed for social choice (Ray [1973]). Having said this, elimination-based independence conditions have also been considered in the context of voting (see e.g., Fishburn [1974]; Young [1995]; and, Ching [1996]).

\(^9\)Apesteguia et al. [2014] call this condition *Condorcet Loser Consistency*. That name usually refers to a much weaker property which stipulates that a Condorcet loser cannot be chosen. To avoid potential confusion, I depart from their nomenclature.
or, Barberà and Gerber [2016]) focus on Euro-Latin and Anglo-American agendas. But, some others focus on specialized agenda formats that are related to tournament solutions\textsuperscript{10} (Coughlan and Le Breton [1999]; Fischer et al. [2011]; Iglesias et al. [2014]) or legislative voting (Banks [1989]; Miller [1995]; Schwartz [2008]).

At the other extreme, a small number of papers are concerned with general conditions which are either necessary (McKelvey and Niemi [1978]; Moulin [1986]; Srivastava and Trick [1996]) or necessary and sufficient (Horan [2013]) for implementation by agenda. On the whole, the existence results in these papers are non-constructive. So, while they delimit what outcomes can and cannot be achieved with strategic agenda voting in general, they are silent about what outcomes can be achieved with a particular agenda format.

The current paper bridges the gap between these two strands of the literature — by characterizing the relationship between “form” and “function” for a wide range of agendas used in legislative settings. Since the prior work on this issue is somewhat limited, the main connection to the literature is methodological.

In this regard, two papers merit particular emphasis. The first is Ordeshook and Schwartz [1987], which proposes a useful taxonomy of agenda features (later extended by Miller [1995]). While I do not rely on their taxonomy, my work is motivated by a similar desire to abstract away from institutional details and focus on the key structural elements that typify legislative agendas. The second is Apesteguia et al. [2014], which introduces a decision rule framework to analyze agenda voting.\textsuperscript{11} Unlike the conventional approach to implementation, this framework (which I adopt in the current paper) allows both the voter preferences and the set of alternatives to vary. For agendas, this does not enrich the scope of implementable rules (see Remark 1); but, it does simplify considerably the statement and interpretation of the conditions for implementation. As an added bonus, it is also provides a natural framework to study a variety of applications (discussed in Section 7).

2 Definitions

In this section, I review the basic definitions used in the paper.\textsuperscript{12} There is a finite set $N$ of voters and a finite universe $X$ of social alternatives. For convenience, I assume that $|N|$ is odd and that $|N|, |X| \geq 2$. A profile of voter preferences $P \equiv (P_i)_{i \in N}$ is an $|N|$-tuple of (linear) orderings $P_i$ over $X$; an issue $A$ is a non-empty subset of $X$; and, a decision problem is pair $(P, A)$. Let $P$ denote the collection of all profiles; $X$ the collection of all (non-empty) issues; and, $X_j \equiv \{A \in X : |A| = j\}$ the sub-collection of issues with cardinality $j$. Then:

**Definition 1** A decision rule is a mapping $v : P \times X \to X$ such that $v(P, A) \in A$ for all $(P, A) \in P \times X$.

In words, a decision rule prescribes a social outcome for every possible combination of voter preferences and social alternatives. In the sequel, I study the implementation of decision rules by agenda. Formally:

**Definition 2** An agenda $T_A$ on $A \subseteq X$ is a finite binary tree such that:

(i) each terminal node is labeled by (a set consisting of) one alternative in $A$;

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\textsuperscript{10} For a nice overview of the vast literature on tournament solutions, see Laslier [1997] or Brandt et al. [2016].

\textsuperscript{11} The Supplementary Appendix provides a more detailed discussion of the connection to this paper.

\textsuperscript{12} Whenever possible, I follow the notation used by Apesteguia et al. [2014] (see Section 2 of their paper).
(ii) each alternative in \(A\) labels one or more terminal nodes; and,

(iii) each non-terminal node is labeled by the set of alternatives that label its two successors.\(^{13}\)

An agenda defines a game tree: each terminal node (or outcome) represents a social alternative; and, each non-terminal node (or stage game) represents a majority vote between the two subgames that follow it. In legislative settings, each vote relates to a question of policy. By convention, the left subgame represents majority support for “aye” while the right subgame represents majority support for “nay.” With this interpretation in mind, the first two features of Definition 2 ensure that: (i) every sequence of questions leads to some social outcome; and, (ii) every social outcome results from some sequence of questions. In turn, feature (iii) amounts to a labeling convention: every node is designated by the outcomes that are reachable from it.\(^{14}\)

To get a sense of these three features, it is worth re-visiting Figures 1 to 4 of the Introduction. There, I abuse notation by omitting the set brackets for the labels of the terminal nodes. In the sequel, I perpetuate this abuse by omitting the brackets for singleton sets when there is no danger of causing confusion.

To adapt an agenda \(T_X\) to a sub-issue \(A \subset X\), one can prune away all of the terminal nodes labeled by infeasible alternatives (see Xu and Zhou [2007]; Horan [2011]; and, Bossert and Sprumont [2013]).

**Definition 3** Given an agenda \(T_X\) on \(X\), the pruned agenda \(T_{X|A}\) on \(A \subseteq X\) is constructed as follows:

1. First, remove every terminal node of \(T_X\) that is labeled by an alternative \(x \in X \setminus A\).
2. Then, remove every node with one successor, connecting its successor to its predecessor.\(^{15}\)
3. Finally, relabel every non-terminal node of the resulting tree to conform with Definition 2.

Like a single-elimination competition in sports, the idea is that the infeasible alternatives (i.e., the alternatives in \(X \setminus A\)) “forfeit” their position without otherwise modifying the agenda. To illustrate, consider the following agenda \(T_X\) on \(X = \{a, a', b, c, x\}\) and the associated pruned agenda \(T_{X|A}\) on \(A = \{b, c, x\}\):

![Diagram](image.png)

Figure 5: **Left**: an agenda \(T_X\). **Right**: the associated pruned agenda \(T_{X|A}\).

\(^{13}\)For a primer of the graph theory terminology used here and elsewhere, see Section 9.1 of the Mathematical Appendix.

\(^{14}\)This convention follows the Farquharson-Miller definition rather than the Ordeshook-Schwartz definition (see Schwartz [2008]). Since my focus is sophisticated rather than sincere voting, the labeling convention plays no role in the analysis. I follow it only because it simplifies the definition of the agenda classes considered in the sequel.

\(^{15}\)In case the root node (which has no predecessor) has a unique successor, its successor becomes the new root node.
Given an agenda $T_A$, each pair $(T_A; P)$ defines a complete information extensive-form game with voter preferences given by $P$ and outcomes in $A$. For games like this, the natural solution concept is Farquharson’s [1969] notion of “sophisticated” voting, which is based on the idea that voters are forward-looking. For this solution concept, the (unique) equilibrium outcome only depends on the majority relation.

**Definition 4** For $(P, A) \in P \times X$, the majority relation $M_A^P$ between alternatives $x, y \in A$ is given by:

$$xM_A^Py \text{ if } \{i \in N : xP_iy\} > |N|/2.$$ 

Since $|N|$ is odd, $M_A^P$ is a total (for distinct $x, y \in A$, $xM_A^Py$ or $yM_A^Px$) and asymmetric (for $x, y \in A$, not $xM_A^Py$ and $yM_A^Px$) binary relation on $A$. In the literature, such a relation is known as a tournament.\(^{16}\)

In any vote at the final stage of the agenda $T_A$, each voter has a weakly dominant strategy to endorse her preferred alternative. So, $M_A^P$ determines which alternative wins a majority in any vote at this stage. When deciding how to vote in any subgame at the penultimate stage, the forward-looking voters discount the alternatives that lose at the final stage; and, again perceive the vote as a choice between two alternatives. So, $M_A^P$ also determines which option (“yea” or “nay”) wins a majority at this stage. By extending this reasoning back to the root node, one can use $M_A^P$ to determine the sophisticated voting outcome of $(T_A; P)$.

As shown by McKelvey and Niemi [1978], this “backward induction” reasoning is equivalent to finding the dominance solvable (Moulin [1979]) or undominated Nash equilibrium outcome of $(T_A; P)$.\(^{17}\) To formalize:

**Definition 5** For a game $(T_A; P)$, the sophisticated voting outcome $U{NE}[T_A; P]$ is defined as follows:

1. First, define $T_A^P(1) \equiv T_A$ to be the original agenda.
2. Then, for each $j > 1$, define the agenda $T_A^P(j + 1)$ from $T_A^P(j)$ as follows:
   - In each terminal subgame of $T_A^P(j)$, prune away the loser according to $M_A^P$; and,
   - Relabel the non-terminal nodes of the resulting binary tree to conform with Definition 2.
3. Finally, define $U{NE}[T_A; P] \equiv T_A^P(K)$ where $K$ is the smallest $j$ such that $T_A^P(j) = T_A^P(j + 1)$.

Having defined the game form and solution concept, it remains to formalize the notion of implementation:

**Definition 6** A decision rule $v$ is implementable by agenda if there exists an agenda $T_X$ such that

$$v(P, A) = U{NE}[T_X|A; P]$$

for every decision problem $(P, A) \in P \times X$. In that case, the agenda $T_X$ implements the decision rule $v$.

\(^{16}\)McGarvey [1953] was the first to consider the problem of finding an upper bound on the minimum number of voters $|N|$ such that $\{M_A^P : P \in P\}$ coincides with the set of all tournaments on $A$. He showed that $|N| \leq |A|^2 - |A|$. Subsequently, Stearns [1959] improved the bound to $|N| \leq |A| + 2$ for $|A|$ even; and, $|N| \leq |A| + 1$ for $|A|$ odd. More recently, Fiol [1992] improved it to $|N| \leq |A| + \lfloor \log_2 |A| \rfloor$ for $|A|$ even; and, $|N| \leq |A| - \lfloor \log_2 |A| \rfloor$ for $|A| - \lfloor \log_2 |A| \rfloor$ odd.

\(^{17}\)The quotation marks (which I continue to use throughout the paper) serve as a reminder that this is not technically equivalent to backward induction. The issue is the lack of full information: every stage game involves simultaneous choices by all voters.
Despite appearances, this is no more general than the usual notion of implementation with a fixed issue $X$. To see this, consider a decision problem $(P, A)$ and let $P^A$ denote a profile that coincides with $P$ on $A$ but, in each voter preference, places the alternatives in $X \setminus A$ (in a specific order) below every alternative in $A$. For an agenda, this “demotion” amounts to the same thing as “pruning away” the alternatives in $X \setminus A$:

**Remark 1** If a decision rule $v$ is implementable by agenda, then $v(P, A) = v(P^A, X)$ for all $(P, A) \in P \times X$.

Put differently, the sub-issues play no role in the analysis of agenda implementation. Once the agenda-setter determines what to implement for $X$, the outcomes for all sub-issues $A \subset X$ are pinned down.

### 3 Simple Agendas

After defining simple agendas, I characterize the family of decision rules implemented by agendas in this class; and, provide a “recipe” for the unique simple agenda that implements each decision rule in this family.

#### 3.1 Definitions and Examples

Simple agendas are defined by two features. The first stipulates that the label of each node entirely determines the structure of the agenda below. For an agenda $T_X$, let $\ell(q)$ denote the label of node $q$. By the convention in Definition 2(3), $\ell(q)$ is the set of outcomes that are reachable from $q$. Letting $q_1$ and $q_2$ denote the two successors of a non-terminal node $q$, the first feature may then be stated as follows:

**Feature 1** An agenda is **history-independent** if, for every pair of non-terminal nodes $q$ and $\bar{q}$,

$$\ell(q) = \ell(\bar{q}) \text{ only if } \{\ell(q_1), \ell(q_2)\} = \{\ell(\bar{q}_1), \ell(\bar{q}_2)\}.$$ 

This condition requires the same question to be posed at any two nodes that ultimately lead to the same set of potential outcomes. Equivalently, the structure of the agenda below each node $q$ can only depend on the outcomes reachable from $q$ (and not on the history of nodes leading to $q$). Practically speaking, this requirement only rules out agendas that include procedural questions — like the right-hand agenda of Figure 5, where the root node amounts to a “question about how to vote” on the issue $A$.

In turn, the second feature requires that outcomes are “contested” in an orderly fashion. Given a non-terminal node $q$, let $u(q) \equiv \ell(q_1) \cap \ell(q_2)$ denote the set of *uncontested* outcomes (that do not risk elimination) at $q$; and, let $c(q) \equiv \ell(q) \setminus u(q)$ denote the set of *contested* outcomes (that do risk elimination) at $q$. Using these definitions, the second feature may be stated as follows:

**Feature 2** An agenda is **persistent** if, for every non-terminal node $q$ such that $u(q) \neq \emptyset$ and each terminal node $t$ below $q$, there exists a non-terminal node $q^*$ between $q$ and $t$ such that $u(q) \in \{\ell(q^*_1), \ell(q^*_2)\}$.

---

18 Given the finiteness of the agenda, this implies Farquharson’s Axiom II: for each non-terminal node $q$, $\ell(q) \neq \ell(q_1), \ell(q_2)$. 

11
Intuitively, the idea is that $u(q)$ represents an issue that must be addressed before making a final choice. Since $q$ does not engage this issue, there must be a subsequent question $q^t$ that does so for each outcome $\ell(t)$ below $q$. This rules out agendas like the left-hand agenda in Figure 5, where the issue $A$ is uncontested at the root $X$ but never addressed thereafter. The problem is that both successors ($A \cup a$ and $A \cup a'$) “mix” the disposition of the issue $A$ with the question of eliminating a previously contested alternative ($a$ or $a'$).

When an agenda is persistent and history-independent, its structure is particularly simple (see Claim 3 of the Appendix). In that case, the question $q^t$ pits $u(q)$ against (an issue that includes) $c(q) \cap \ell(q^t)$. So, $q^t$ either eliminates some of the outcomes uncontested at $q$; or, all of the remaining outcomes contested at $q$. In fact, $q^t$ must be the first question after $q$ where any of the outcomes in $u(q)$ are contested. For each question between $q$ and $q^t$, all of the contested outcomes must have been previously contested at $q$.

**Class 1** An agenda is **simple** if it is both history-independent and persistent.

In the sequel, I denote a simple agenda on $A$ by $S_A$; and, the class of simple agendas (on issues $A \in X$) by $S$. This class includes a wide range of agendas used in practice. Some examples will help to illustrate.

**Example 1** One important sub-class are the **priority agendas** studied in Section 5. For $|X| = 3$, the only priority agendas are the Euro-Latin and Anglo-American agendas. For $|X| = 4$, there are four additional priority agendas (up to permutation): the two agendas in Figure 3; and, the two agendas depicted below.

![Diagram of priority agendas](image)

**Example 2** Another important sub-class are the **knockout agendas** where each alternative labels exactly one terminal node. Up to permutation of the alternatives, there are two knockout agendas on $|X| = 4$ alternatives: the balanced (or Plott-Levine [1976]) agenda below, and, the unbalanced (Euro-Latin) agenda in Figure 2.

---

19 Example 13 of the Supplementary Appendix shows that these two features are independent.

20 It is worth pointing out that a Plott-Levine agenda with $n \geq 1$ questions has $|X| = 2^n$ alternatives.
Example 3 When \(|X| = 3\), every simple agenda is either a knockout agenda or a priority agenda. As the number of alternatives grows, an increasing variety of simple agendas fall outside these two sub-classes. When \(|X| = 4\), for instance, there are already two such agendas (up to permutation):

Examples 2-3 describe all of the simple agendas on \(|X| = 4\) alternatives: the six priority agendas in Example 1; the Plott-Levine agenda in Example 2; and, the two agendas in Example 3. One natural way to build larger agendas from these nine agendas is through the formation of compound agendas.

**Definition 7** Given an agenda \(T_A\) on \(A = \{a_1, \ldots, a_m\}\) and a collection of agendas \(\{T_{A_i}^{a_i} : a_i \in A\}\) such that \(D_i \cap A = a_i\) for each \(i\), the compound agenda \(T_A \otimes \{T_{A_i}^{a_i} : a_i \in A\}\) on \(D = \bigcup_{a_i \in A} D_i\) is constructed as follows:

1. First, replace each terminal node of \(T_A\) labeled by \(a_i \in A\) with the agenda \(T_{A_i}^{a_i}\).
2. Then, relabel the non-terminal nodes of the resulting binary tree to conform with Definition 2.

The idea is that \(D \setminus A\) consists of alternatives that were not up for consideration in \(T_A\). Instead of having to settle on \(a_i \in A\), the voters have the opportunity to select among the alternatives in \(D_i\). To do so, they follow the agenda \(T_{A_i}^{a_i}\) at any point where they would have otherwise selected \(a_i\) from \(T_A\).

It is clear that the simplicity of the “base” agenda \(T_A\) and every “extension” \(T_{A_i}^{a_i}\) is necessary but not sufficient for the resulting compound agenda to be simple. Nonetheless, simplicity is preserved under a variety of natural compounding procedures that arise in practice — like the procedure used to extend priority agendas (discussed in Example 9 below); or, the procedure used to “standardize” an agenda.
\section*{Definition 8} An agenda is **standard** if the last question always involves the **status quo** alternative \(x^*\). Clearly, one can “standardize” any agenda \(T_A\) such that \(x^* \notin A\) by adding a final question at each terminal node. Formally, this defines a compound agenda \(T_A^* \equiv T_A \otimes \{S_{\{a_i,x^*\}} : a_i \in A\} \) on \(A \cup x^*\).

It is not difficult to see that simplicity is preserved under standardization. In fact:

**Remark 2** A standardized agenda is simple if and only if its base agenda is simple.

It would appear that standardized simple agendas are fairly common in practice.

**Example 4** Among the agendas in the current paper, the left-hand agenda in Figure 3 is a standardized Euro-Latin agenda; and, the one in Figure 4 is a standardized Plott-Levine agenda. (For these agendas, \(x_4\) is the status quo.) Some additional examples can be found in other papers: the agenda in Figure 14 of Ordeshook and Schwartz [1987] is a standardized version of an eight-alternative Plott-Levine agenda; the agenda in their Figure 5 is a standardized priority agenda (which extends the left-hand agenda in Example 1); and, the agenda in Krehbiel and Rivers [1990] is a standardized version of an unbalanced knockout agenda on six-alternatives.

### 3.2 Implementation

The necessary and sufficient conditions for implementation by simple agenda are two natural conditions related to Path Independence and the Independence of Irrelevant Alternatives (IIA).

The first condition stipulates that the outcome can be determined by “splitting up” an issue into simpler issues. A pair \((B, C) \in X \times X\) defines a **splitting** of \(A \in X\) for a decision rule \(v\) if:

(i) \(B \cup C = A\) (so that \(B\) and \(C\) form an exact bi-cover of \(A\));

(ii) \(B \cap C \neq B, C\) (so that \(B\) and \(C\) are non-nested sets); and,

(iii) \(v(P, A) = v(P, \{v(P, B), v(P, C)\})\) for every profile \(P \in P\).

When there exists a splitting of \(A\), \(v\) is **splittable** for \(A\). Then, the first condition can be stated as follows:

**Issue Splitting (IS)** For every issue \(A \notin X_1 \cup X_2\), \(v\) is splittable.

By comparison, Path Independence imposes a much stronger requirement: the equality in (iii) must hold for all pairs \((B, C)\) that (i) form an exact bi-cover of \(A\) even when (ii) the sub-issues \(B\) and \(C\) are nested. Likewise, IS weakens the **Division Consistency** condition of Apesteguia et al. [2014] (see footnote 7 above).\(^{21}\) The main difference is that (ii) does not require the sub-issues \((B, C)\) of \(A\) to be disjoint. Another difference is that (iii) does not impose any consistency between the splitting of \(A\) and its sub-issues \(D \subset A\).

In turn, the second condition stipulates that outcomes cannot be affected by the presence of a universally unappealing alternative. An alternative \(a \in A\) is the **Condorcet loser** for a decision problem \((P, A) \in P \times X\) if \(xM^*_{\alpha}a\) for all \(x \in A \setminus a\). Then, a decision rule \(v\) is **independent of the losers** for \(A \in X\) if: for all \(P \in P\) and

---

\(^{21}\)Division Consistency may be stated as follows. For every issue \(A \notin X_1 \cup X_2\), there exists a pair \((B, C) \in X \times X\) such that:

(i') \(B \cup C = A\); (ii') \(B \cap C = \emptyset\); and, (iii') \(v(P, D) = v(P, \{v(P, B \cap D), v(P, C \cap D)\})\) for each profile \(P \in P\) and each \(D \subseteq A\).
a \in A, \nu(P, A) = \nu(P, A \setminus a)\) if \(a\) is the Condorcet loser for \((P, A)\). The second condition requires this kind of independence for every non-degenerate issue.

**Independence of Losing Alternatives (ILA)** For every issue \(A \notin X_1\), \(\nu\) is independent of the losers.

By comparison, IIA imposes a much stronger requirement: the removal of any unchosen alternative must not affect the outcome, regardless of whether this alternative is the Condorcet loser.

Together, these two conditions characterize implementation by simple agenda. As shown in Section 3.3 below, the structure of the unique implementing agenda is determined by the outcomes of \(\nu\) for profiles, called *Condorcet triples*, where the majority preference relation forms a cycle on three alternatives:

**Theorem 1 (Necessary and Sufficient Conditions)** A decision rule \(\nu\) is implementable by a simple agenda if and only if it satisfies Issue Splitting and the Independence of Losing Alternatives. **(Uniqueness)** For every decision rule \(\nu\) that satisfies these two properties, there is a unique simple agenda \(S'X\) that implements \(\nu\); and, the structure of \(S'X\) is fully determined by the outcomes on Condorcet triples.

The Supplementary Appendix contains examples which establish that the conditions IS and ILA are independent. In general, ILA is necessary for implementation by any kind of agenda (whether or not the agenda in question is simple). This is a straightforward consequence of Remark 1: "backward induction" is unaffected by pruning the Condorcet loser away from an agenda. Conversely, IS is not generally necessary for implementation by agenda, as the following example shows. To simplify the presentation, let \(P_{xyz}\) denote a Condorcet triple where \(x\) is majority preferred to \(y\), \(y\) to \(z\), and \(z\) to \(x\).

**Example 5** The decision rule implemented by the agenda \(T_X\) in Figure 5 violates IS. To see this, it is enough to observe that, for the decision problems \((P_{xbc}, A)\) and \((P_{xcb}, A)\), the outcome is the majority winner between \(b\) and \(c\). So, \(\text{UNE}[t_X; P_{xbc}] = b\) cannot be paired with \(x\) in any potential splitting of \(A\). Otherwise, \(b\) must be eliminated by \(x\), which is majority preferred on \((P_{xbc}, A)\). By the same reasoning, \(\text{UNE}[t_X; P_{xcb}] = c\) cannot be paired with \(x\). This leaves \((x, \{b, c\}\) as the only potential splitting of \(A\). Since that would require \(x\) as the outcome for both \((P_{xbc}, A)\) and \((P_{xcb}, A)\) however, there can be no way to split \(A\).

### 3.3 A Recipe

For a decision rule \(\nu\) that satisfies IS and ILA, every issue \(A \notin X_1\) has a unique splitting (see Claim 8 of the Appendix). This makes it straightforward to define, by recursion, the unique simple agenda that implements \(\nu\). To formalize, let \(S_{\{\ell_1, \ell_2\}}\) denote the unique simple agenda with two terminal nodes labeled \(\ell_1\) and \(\ell_2\); and, let \(S_{\{\ell\}}\) denote the unique degenerate agenda with a single node labeled \(\ell\).

**Definition 9** Given a decision rule \(\nu\) that satisfies IS and ILA, the agenda \(S'X\) is defined as follows:

1. First, define \(S(1) = S_{\{X\}}\) to be the degenerate agenda whose only node is labeled \(X\).

---

22Since \(|N| \geq 3\), it is guaranteed that such a profile \(P_{xyz}\) exists (see footnote 16 above).
(2) Then, for each \( j > 1 \), define the agenda \( S^v(j + 1) \) from \( S^v(j) \) as follows:
- For each terminal node labeled by a non-singleton \( A \) (if any), determine its splitting \((B_A, C_A)\); and,
- Replace every terminal node of \( S^v(j) \) labeled by \( A \) with the simple agenda \( S_{\{B_A, C_A\}} \).

(3) Finally, define \( S^v(X) \equiv S^v(|X| - 1) \).

Since the recursion continues for \(|X| - 1\) steps, each terminal node of \( S^v_X \) must be labeled by a singleton. It follows that \( S^v_X \) defines an agenda. What is more, this agenda is history-independent: at any interim stage, the construction is identical for each terminal node labeled by the issue \( A \). While not immediately apparent from Definition 9, it turns out that the agenda \( S^v_X \) is also persistent (see Claim 9 of the Appendix).

![Figure 6: The recursive construction of \( S^v_X \).](image)

In Figure 6, the leftmost nodes below \( B_X \) illustrate the construction for \(|A| = 2\) while the leftmost nodes below \( C_X \) illustrate it for \(|A| > 2\). In turn, the two triangles represent the subgames starting from the nodes labeled \( C_{B_X} \) and \( C_{C_X} \) while the ellipses indicate where details have been omitted.23

As Theorem 1 indicates, Condorcet triples may be used to describe \( S^v_X \) more explicitly. To see how, fix an issue \( \{x, b, c\} \). Clearly, there are three potential splittings where \( b \) and \( c \) appear in separate sub-issues:

\[
(\{b, x\}, \{c, x\}) \quad (b, \{c, x\}) \quad (\{b, x\}, c)
\]

Each of these splittings corresponds to the initial stage game of a different simple agenda on \( \{x, b, c\} \):

---

23It is important not to confuse the agenda \( S_X(A) \) with the agenda \( S_{X|A} \). While the former refers to the subgame at node \( A \) in \( S_X \), the latter refers to the pruned agenda on \( A \) obtained from \( S_X \). As Figure 5 illustrates, these agendas may be quite different.
In turn, each of these agendas implements a different combination of outcomes on the two Condorcet triples:

![Figure 7: Simple agendas on three alternatives \( \{x, b, c\} \).](image)

<table>
<thead>
<tr>
<th>Profile ( \backslash ) Agenda</th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_{xbc} )</td>
<td>( b )</td>
<td>( b )</td>
<td>( c )</td>
</tr>
<tr>
<td>( P_{xcb} )</td>
<td>( c )</td>
<td>( c )</td>
<td>( b )</td>
</tr>
<tr>
<td>Outcomes</td>
<td>Majority loser between ( b ) and ( c )</td>
<td>Outcome ( b ) for both triples</td>
<td>Outcome ( c ) for both triples</td>
</tr>
</tbody>
</table>

Table 1: Outcomes implemented by agendas (a)-(c) on \( \{x, b, c\} \).

By construction, there exist alternatives \( b_A \) and \( c_A \) that appear only on opposite sides of the agenda \( S_X^A \) starting at any node labeled \( A \). For these alternatives, the outcomes on each issue \( \{x, b_A, c_A\} \) involving an alternative \( x \in A \) must coincide with one of the possibilities in Table 1. Using this to pin down alternatives \( b_A, c_A \in A \), one can then use Table 1 to describe the unique splitting \( (B_A, C_A) \) of \( A \) in terms of Condorcet triples:

\[
B_A \equiv \{b_A\} \cup \{x \in A \setminus \{b_A, c_A\}\} \text{ : the outcomes on } \{x, b_A, c_A\} \text{ are type-(a) or type-(c)}; \quad \text{and}
\]

\[
C_A \equiv \{c_A\} \cup \{x \in A \setminus \{b_A, c_A\}\} \text{ : the outcomes on } \{x, b_A, c_A\} \text{ are type-(a) or type-(b)}.
\]

Intuitively, the type-(a) outcomes reveal that \( x \) appears in both sub-issues of \( (B_A, C_A) \) while the type-(c) outcomes (resp. type-(b) outcomes) reveal that \( x \) appears only in the same sub-issue as \( b_A \) (resp. \( c_A \)).

### 4 Comparison with Recursive Agendas

In this section, I consider an agenda class that is somewhat broader than the class of simple agendas; and, highlight two appealing properties of simple agendas that do not extend to every agenda in this class.
4.1 Definitions

Despite their prevalence, history-independence and persistence do not figure in the agenda taxonomy proposed by Ordeshook and Schwartz. Instead, these authors identify two other common features of agendas. The first stipulates that one or more alternatives is eliminated at every non-terminal node. Such an agenda is non-repetitive: for every non-terminal node \( q \), \( c_i(q) \equiv c(q) \cap \ell(q_i) \neq \emptyset \) for \( i = 1, 2 \).\(^{24}\) In turn, the second feature requires that some of the contested outcomes at each node continue to be contested until they are eliminated or ultimately selected. Such an agenda is continuous: for every non-terminal node \( q \) such that \( c_i(q) \neq \emptyset \), there exists some alternative in \( c_i(q) \) labels exactly one terminal node below \( q_i \).\(^{25}\)

It is clear that non-repetitiveness and continuity are independent features of agendas (see Example 11 of the Supplementary Appendix). However, they can be re-stated more succinctly in combination:

**Remark 3** An agenda is non-repetitive and continuous if and only if, for every non-terminal node \( q \) and each successor \( q_i \) of \( q \), there exists an alternative \( x_{ij}^q \in c_i(q) \) that labels exactly one terminal node below \( q_i \).

Agendas with these two features have an appealing recursive structure. Where \( \ell_1(T_A) \) denotes the alternatives in \( A \) that label exactly one terminal node in \( T_A \), \( T_X = S_{\{b,c\}} \otimes \{T_B, T_C\} \) is non-repetitive and continuous if and only if: (i) both \( T_B \) and \( T_C \) are non-repetitive and continuous; and, (ii) \( \ell_1(T_B) \cap \ell_1(T_C) \neq \ell_1(T_B), \ell_1(T_C) \) (so that \( \ell_1(T_B) \) and \( \ell_1(T_C) \) are non-nested). This observation motivates the following definition.

**Class 2** An agenda is **recursive** if it is both non-repetitive and continuous.

In the sequel, I denote a recursive agenda on \( A \) by \( R_A \); and, the class of all recursive agendas by \( R \). It is not difficult to show that this class is strictly larger than the class of simple agendas (i.e., \( R \supset S \)):

**Remark 4** (i) Every simple agenda is recursive. (ii) But, not every recursive agenda is simple.

While this observation does not provide a clear sense of the gap between the two classes, there is reason to suspect that it might be significant. For one, many recursive agendas seem to lack one (or both) of the features that define simple agendas (see Example 13 of the Supplementary Appendix). What is more, their number would appear to increase rapidly with the size of \( X \). Already for \( |X| = 4 \), six of the fifteen recursive agendas are non-simple (see Example 12 of the Supplementary Appendix).

4.2 Two Key Distinctions

Despite the apparent gap between the two classes, very few of the recursive agendas used in practice are non-simple. None of the agendas discussed by Ordeshook and Schwartz fit into this category, for instance. One

\(^{24}\)This directly implies Farquharson’s Axiom II (see footnote 18 above).

\(^{25}\)Originally, Ordeshook and Schwartz [1987, p.185] only defined continuity for “Ordeshook-Schwartz” agendas (see footnote 14). Miller [1995, p.26] later adapted the concept to “Farquharson-Miller” agendas by requiring that every contested alternative be contested until it is either eliminated or selected. The underlying motivation for the weaker definition given here is to address the problems pointed out by Groseclose and Krehbiel [1993].
plausible explanation is that voting outcomes on (non-simple) recursive agendas lack the compelling properties of voting outcomes on simple agendas. In this section, I identify two such properties.

The first is a monotonicity property for voter preferences. Given a profile $P$, let $P^{\uparrow x}$ denote a profile where every voter’s preference is identical to her preference in $P$ except for one voter, whose preference between $x$ and the immediately preferred alternative (if any) is reversed. Thus, $P^{\uparrow x}$ differs from the original profile $P$ only by improving the alternative $x$ in the eyes of a single voter. Then, a decision rule $v$ is preference monotonic at $A$ if, for every $P \in P$, $v(P, A) = x$ implies $v(P^{\uparrow x}, A) = x$ for every profile $P^{\uparrow x}$.

**Preference Monotonicity** At every issue $A \notin X_1$, $v$ is preference monotonic.

Clearly, every knockout agenda (induces a decision rule that) satisfies Preference Monotonicity (Moulin [1991]; Altman et al. [2009]). To see this, suppose that $x$ is the outcome at $P$; and, consider the backward induction “path” that leads to $x$. Since each alternative appears only once, what $x$ faces along this path cannot be affected by improving it. So, $x$ is still the outcome at $P^{\uparrow x}$. In fact, the same is true for Anglo-American agendas (Moulin [1986, 1991]). Even though some alternatives now appear more than once, the improvement cannot affect which alternatives the original winner faces. As it turns out, this extends to all simple agendas.

**Proposition 1** Every decision rule $v$ implementable by simple agenda satisfies Preference Monotonicity. Conversely, for each $X$ such that $|X| \geq 5$, there are recursive agendas that violate Preference Monotonicity.

The following example (which extends easily to $|X| \geq 6$) establishes the second part of the statement.

**Example 6** Consider the recursive (but non-persistent) agenda on $X = \{x_1, x_2, x_3, x_4, x_5\}$ depicted below.

\[\begin{array}{c}
\{x_1, x_2, x_3, x_4, x_5\} \\
\{x_1, x_3\} \\
\{x_2, x_3, x_4, x_5\} \\
\{x_3, x_4, x_5\} \\
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{array}\]

To see that this agenda violates Preference Monotonicity, fix a profile $P$ that induces the (partial) majority relation $M_P^P$ shown above.\(^{26}\) It is clear that the sophisticated outcome on $P$ is $x_5$: this alternative is the winner under the “left” branch of the agenda and it defeats $x_3$, the alternative which emerges from the “right” branch.

\(^{26}\)Technically, $|N|$ must be “sufficiently large” relative to $|X|$ (see footnote 16 above). Since $|X| = 5$, it is enough that $|N| \geq 5$. 

---

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When the majority relation between $x_4$ and $x_5$ is reversed (thereby improving $x_5$), the outcome changes: $x_2$ now emerges from the “right” branch and it defeats $x_5$, the winner under the “left” branch.

Since every recursive agenda on $|X| \leq 4$ alternatives satisfies Preference Monotonicity, this counter-example is minimal (both in terms of the number of alternatives and the number of terminal nodes).

The second appealing property of simple agendas relates to implementation on restricted domains.

**Definition 10** Given a decision rule $v$ and an issue $A \in X$, the **restricted** rule $v_A : P \times A \to A$ is defined by

$$v_A(P, D) \equiv v(P, D) \text{ for each } (P, D) \in P \times A.$$  

Pruning provides a simple way to implement restricted rules: if $T_X$ implements $v$, then the pruned agenda $T_{X|A}$ implements $v_A$. One drawback is that this pruning tends to destroy the structural features of the original agenda. To illustrate, consider Figure 5. While the agenda on the left is recursive, the pruned version on the right is not. For a class of agendas $T$ that satisfies the following property, this does not present a problem.

**$T$-Consistency** For every issue $A \in X$, the restricted rule $v_A$ is implementable by an agenda $T_A \in T$.

Rather than using the pruned agenda $T_{X|A}$ to implement $v_A$, one can use an agenda $T_A \in T$. One advantage relates to the possibility of simplifying the voting procedure: pruned agendas tend to include redundant questions that cannot affect the outcome at any profile (like the node $\{c, x\}$ for the right-hand agenda in Figure 5).

Another advantage relates to agenda-setting. Imagine that $T$ represents the agendas permitted by parliamentary procedure; and, $A \subset X$ reflects the “real” issue up for debate. By introducing “dummy” alternatives $B$, one might generate an agenda $T_{A\cup B} \in T$ that differs substantively from the permitted agendas on $A$ (i.e., for each $T_A \in T$, there exists a profile $P \in P$ such that $UNE[T_{A\cup B}|A; P] \neq UNE[T_A; P]$). By restricting oneself to a class of agendas $T'$ that satisfies $T$-Consistency, this type of manipulation can be ruled out.

On either account, the most appealing agendas classes are the “self-consistent” classes. Formally, an agenda class $T$ satisfies **Self-Consistency** if the decision rule induced by every agenda $T_A \in T$ satisfies $T$-Consistency. Trivially, the Euro-Latin agendas exhibit this property: pruning preserves the structure of the agenda. In fact, the Anglo-American agendas do as well (Apesteguia et al. [2014]). More generally, the class of simple agendas satisfies Self-Consistency; but, the class of recursive agendas does not.

**Proposition 2** Every decision rule $v$ implementable by simple agenda satisfies $S$-Consistency. Conversely, for each $X$ such that $|X| \geq 4$, there are decision rules implementable by recursive agenda that violate $R$-Consistency.

The first statement follows from Theorem 1: if $v$ satisfies IS and ILA, then so does the restricted rule $v_A$ on $A \subset X$. (In fact, one can use the recipe in Section 3.3 to define the simple agenda on $A$ that implements $v_A$.) In turn, the following example (which is easily extended to $|X| \geq 5$) establishes the second statement:

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27If a decision rule $v$ on $X$ satisfies the axioms listed in their Theorem 2, then so does the restricted rule $v_A$ on $A \subset X$. 

---
Example 7 Consider the two recursive (but non-persistent) agendas on \(X = \{x_1, x_2, x_3, x_4\}\) depicted below.

The left-hand agenda violates \(\mathcal{R}\)-Consistency. To see this, note that the outcome for each Condorcet triple on \(\{x_2, x_3, x_4\}\) is the majority winner between \(x_2\) and \(x_4\) — which violates Issue Splitting (by Example 5).

Conversely, the right-hand agenda satisfies \(\mathcal{R}\)-Consistency. To see this, it suffices to check that no pair of Condorcet triples has outcomes that coincide with the majority winner between two of the three alternatives.

The left-hand agenda in this example shows that \(\mathcal{R}\) violates Self-Consistency: this agenda is recursive but does not satisfy \(\mathcal{R}\)-Consistency. In turn, the right-hand agenda shows that \(S\) is not the largest self-consistent sub-class of \(\mathcal{R}\): since it satisfies \(\mathcal{R}\)-Consistency, it can be added to \(S\) to make a larger self-consistent sub-class.

5 Priority Agendas

After first defining priority agendas, I then characterize the decision rules implemented by agendas in this class.

5.1 Definitions and Examples

A priority agenda on \(X\) is defined by an admissible pair \((\succeq, \alpha)\) consisting of a priority \(\succeq\) and an amendment rule \(\alpha\). This pair provides instructions for building the agenda: the priority determines when each alternative is added to the agenda; and, the amendment rule determines how each is added to the agenda.

Formally, a priority \(\succeq\) is a weak ordering on \(X\) whose indifference classes contain at most two alternatives. If \(x \succ y\), then \(x\) has higher priority and is added to the agenda before \(y\). (If \(x \sim y\), then \(x\) and \(y\) have equal priority and are added simultaneously.) Let \(\tilde{X}_j\) denote the \(j^{th}\) highest indifference class of \(\succeq\) (so that \(\tilde{X}_1\) denotes the set of top priority alternative(s) in \(X\) and \(X \setminus \tilde{X}_1\) the set of non-top priority alternative(s) in \(X\)).

In turn, an amendment rule \(\alpha\) is a mapping \(\alpha : X \to X_2 \cup X_1 \cup \emptyset\) that assigns at most two alternatives to each alternative \(z \in X\). The interpretation is that \(\alpha(z)\) is the (set of) highest priority alternative(s) that are faced (or "amended") by \(z\). Formally, one alternative \(z \in X\) amends another alternative \(y \in X\) if

\[
y \in \alpha(z) \quad \text{or} \quad x \succ y \succ z \quad \text{for some} \quad x \in \alpha(z).
\]
In other words, \( z \) amends each \( x \in \alpha(z) \) and each \( y \) that is added to the agenda between \( x \) and \( z \).

Finally, to be admissible, a pair \( (\succ, \alpha) \) must satisfy the following conditions for each \( z \in X \):

(i) if \( z \notin \bar{X}_1 \), then \( \alpha(z) = \emptyset \);
(ii) if \( z \in \bar{X}_j \) and \( j \neq 1 \), then \( \alpha(z) \subseteq \bar{X}_k \) for some \( k < j \);
(iii) if \( z, z' \in \bar{X}_j \) (and \( j \neq 1 \)), then \( \alpha(z) = \alpha(z') \); and,
(iv) if \( z \in \bar{X}_{j+1} \) and \( |\bar{X}_j| = 2 \), then \( \alpha(z) = \bar{X}_j \) or \( \alpha(z) \subseteq \bar{X}_k \) for some \( k < j \).

These conditions ensure that a pair \( (\succ, \alpha) \) is consistent with the desired interpretation of the parameters. In words, they stipulate that: (i) top priority alternatives amend nothing; (ii) non-top priority alternatives amend higher priority alternative(s); (iii) equal priority alternatives amend the same alternative(s); and, (iv) an alternative with priority immediately below two equal priority alternatives amends both.

To re-construct a priority agenda from an admissible pair \( (\succ, \alpha) \), one uses the amendment rule \( \alpha \) to add the alternatives in decreasing order of priority \( \succ \). To formalize this recursive construction:

**Class 3** Given an admissible pair \((\succ, \alpha)\) on \(X\), the priority agenda \(Pr_{(\succ, \alpha)}\) is defined as follows:

1. First, define \(Pr_{(\succ, \alpha)}(1) = S_{\bar{X}_1}\) to be the simple agenda on the top priority alternative(s) \(\bar{X}_1\) of \(\succ\).

2. Then, for each \( j > 1 \), define \(Pr_{(\succ, \alpha)}(j + 1)\) from \(Pr_{(\succ, \alpha)}(j)\) as follows:
   (i) If \( x_k \in \alpha(x_{j+1}) \cap \bar{X}_k \), replace every terminal node of \(Pr_{(\succ, \alpha)}(j)\) labeled by:
       - \( x_k \) with the simple agenda \(S_{\{x_k, \bar{X}_{j+1}\}}\); and,
       - \( x_{k'} \in \bar{X}_{k'} \) such that \( k < k' \leq j \) with the simple agenda \(S_{\{x_{k'}, \bar{X}_{j+1}\}}\).
   (ii) If \( |\bar{X}_{j+1}| \neq 1 \), replace every terminal node \(X_{j+1}\) resulting from (i) with the simple agenda \(S_{\bar{X}_{j+1}}\).

3. Finally, define \(Pr_{(\succ, \alpha)} = Pr_{(\succ, \alpha)}(K)\) where \( K \equiv |X/\sim| \) is the number of indifference classes in \(\succ\).

At any stage \( j \leq K \) of this recursion, \(Pr_j^{(\succ, \alpha)}\) is compounded (see Definition 7) by adding simple agendas (consisting of two or three alternatives) to some of the terminal nodes. To illustrate this construction:

![Figure 8: Detail at the terminal node \(x_{k'}\) in stage \(j \leq K\) of the construction.](image)

---

28Technically, one must then relabel the non-terminal nodes to conform with Definition 2. Since this step is straightforward but cumbersome, it has been omitted to preserve clarity. For the details, please see Claim 21 of the Appendix.

29Technically, it is not necessary to add equal priority alternatives simultaneously. Nothing would change by adding them one at a time (provided that one re-interpreted the amendment rule to mean that the second of the two additions only amends the first).
This diagram makes it clear that the resulting agenda must be history-independent. In fact, it must also be persistent (see Claim 23 of the Appendix). In other words, every priority agenda must be simple. At the same time, not every simple agenda is a priority agenda (see Example 3).

**Remark 5** (i) Every priority agenda is simple. (ii) But, not every simple agenda is a priority agenda.

To simplify the notation, I sometimes suppress the priority-amendment pair \((\succsim, \alpha)\) and simply denote a priority agenda on \(X\) by \(\mathcal{P}_X\). I also denote the class of all priority agendas by \(\mathcal{PR}\).

The following examples highlight some important features of priority agendas. The first shows that Euro-Latin and Anglo-American agendas are easy to represent in terms of priority-amendment pairs.

**Example 8** Euro-Latin and Anglo-American agendas on \(X = \{x_1, ..., x_m\}\) both admit two representations:

<table>
<thead>
<tr>
<th>Agenda</th>
<th>Representation</th>
<th>Priority (\succsim)</th>
<th>Amendment Rule (\alpha)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euro-Latin</td>
<td>strict</td>
<td>(x_1 \succ ... \succ x_{m-1} \succ x_m)</td>
<td>(\alpha(x_i) = x_{i-1}) for (2 \leq i \leq m)</td>
</tr>
<tr>
<td></td>
<td>weak</td>
<td>(x_1 \succ ... \succ x_{m-1} \sim x_m)</td>
<td>(\alpha(x_i) = \begin{cases} x_{i-1} &amp; \text{for } 2 \leq i \leq m-2 \ x_{m-2} &amp; \text{for } i = m-1, m \end{cases})</td>
</tr>
<tr>
<td>Anglo-American</td>
<td>strict</td>
<td>(x_1 \succ x_2 \succ ... \succ x_m)</td>
<td>(\alpha(x_i) = x_1) for (2 \leq i \leq m)</td>
</tr>
<tr>
<td></td>
<td>weak</td>
<td>(x_1 \sim x_2 \succ ... \succ x_m)</td>
<td>(\alpha(x_i) = {x_1, x_2}) for (3 \leq i \leq m)</td>
</tr>
</tbody>
</table>

The non-uniqueness in the representation of both agendas is due to symmetry between alternatives (\(x_{m-1}\) and \(x_m\) on the Euro-Latin agenda; and, \(x_1\) and \(x_2\) on the Anglo-American agenda). For the symmetric alternatives in both agendas, one can either assign equal priority or consecutive priority. In fact, this is true for all priority agendas with symmetric alternatives. Having said this, the problem of non-uniqueness is not limited to agendas with symmetric alternatives: the right-hand agenda in Example 1 admits two representations even though it contains no symmetric alternatives (see Example 14 of the Supplementary Appendix for the details).\(^{30}\)

In turn, the second example shows how to extend priority agendas by compounding (see Definition 7):

**Example 9** Up to permutation, the only priority agendas on \(\{x_1, x_2, x_3\}\) are the (i) Euro-Latin and (ii) Anglo-American agendas in Figure 1. By starting from a strict priority on \(\{x_1, x_2, x_3\}\), there are three distinct ways to extend each of these agendas into a compound agenda on \(\{x_1, x_2, x_3, x_4\}\).

\(^{30}\)It is worth pointing out that one cannot eliminate the problem of multiplicity by requiring a representation to use strict priority. The issue is that some priority agendas cannot be represented with a strict priority (see Example 15 of the Supplementary Appendix). Instead, the way to eliminate multiplicity is to allow strict priority only when it is necessary to represent the agenda.
- **Euro-Latin amendment**: \( x_4 \) amends only the lowest priority alternative \( x_3 \) in \( \{x_1, x_2, x_3\} \). This leads to (i) the Euro-Latin agenda in Figure 2 and (ii) the right-hand agenda in Figure 3.

- **Anglo-American amendment**: \( x_4 \) amends every alternative in \( \{x_1, x_2, x_3\} \). This leads to (i) the left-hand agenda in Figure 3 and (ii) the Anglo-American agenda in Figure 2.

- **Intermediate amendment**: \( x_4 \) amends the two lowest priority alternatives \( x_2 \) and \( x_3 \) in \( \{x_1, x_2, x_3\} \) (but not \( x_1 \)). This type of amendment leads to the two agendas depicted in Example 1.

By starting from a weak priority on \( \{x_1, x_2, x_3\} \), one cannot generate any additional agendas. As a result, the six agendas listed above exhaust the collection of priority agendas on four alternatives (up to permutation).

### 5.2 Implementation

Euro-Latin and Anglo-American agendas have the property that the lowest priority alternative is chosen only when it is unambiguously appealing. To formalize, an alternative \( a \in A \) is the *Condorcet winner* for the decision problem \((P, A) \in \mathcal{P} \times \mathcal{X}\) if \( a \in M_x^P \) for all \( x \in A \setminus a \). For a decision rule \( v \), an alternative \( a^* \in A \) is *marginal* on \( A \in \mathcal{X} \) if, for all \( P \in \mathcal{P} \), \( v(P, A) = a^* \) only if \( a^* \) is the Condorcet winner for \((P, A)\). Then, the stated property of Euro-Latin and Anglo-American agendas can be formalized as follows:

**Marginalization** (M) *On every issue \( A \notin X_1 \cup X_2 \), \( v \) has a marginal alternative.*

The next result shows that this property is satisfied by every priority agenda.

**Proposition 3** *Every decision rule \( v \) that is implementable by priority agenda satisfies Marginalization.*

In a sense, M is the distinguishing property of decision rules that are implementable by priority agenda.

**Theorem 2** *(Necessary and Sufficient Conditions)* *A decision rule \( v \) is implementable by a priority agenda if and only if it satisfies Issue Splitting, the Independence of Losing Alternatives, and Marginalization.* *(Uniqueness)* *For every decision rule \( v \) satisfying these properties, there is a unique priority agenda \( \Pr^v_X \) that implements \( v \); and, the pair \((\succ^v, \alpha_v)\) that defines \( \Pr^v_X \) is fully determined by the outcomes on Condorcet triples.*

To establish sufficiency, a key step is to determine the priority structure associated with any decision rule that satisfies the axioms. To do this, I rely on the familiar tool of revealed preference.

**Definition 11** *For a decision rule \( v \), first define the binary relations \( \succ^v \) and \( \sim^v \) on \( \mathcal{X} \) as follows:

- \( y \succ^v z \) if there exists an issue \( A \in \mathcal{X} \) such that \( y \in A \) where \( z \) is marginal and \( y \) is not; and,
- \( y \sim^v z \) if, for every issue \( A \in \mathcal{X} \) such that \( y, z \in A \), \( y \) is marginal if and only if \( z \) is marginal.

Then, define the revealed priority \( \succeq^v \) on \( \mathcal{X} \) by \( y \succeq^v z \) if \( y \succ^v z \) or \( y \sim^v z \).*
For the Anglo-American and Euro-Latin agendas, it is easy to see that revealed priority reflects the “true” (underlying) priority. For every issue, the Anglo-American agenda marginalizes the lowest ranked alternative in \( A \) according to \( \succcurlyeq \) while the Euro-Latin agenda marginalizes the two lowest ranked alternatives. In fact, \( \succcurlyeq \) defines a priority with these kinds of features for any decision rule that satisfies the axioms:

**Lemma 1** If a decision rule \( v \) satisfies IS, ILA, and M, then: (i) the revealed priority \( \succcurlyeq \) defines a priority (i.e., \( \succcurlyeq \) is a complete and transitive binary relation whose indifference classes contain at most two alternatives); and; (ii) for each issue \( A \in X \), \( v \) marginalizes the lowest or two lowest alternatives in \( A \) according to \( \succcurlyeq \).

As Theorem 2 indicates, the revealed priority \( \succcurlyeq \) can be re-defined entirely in terms of Condorcet triples. Table 1 suggests how to do this. For each issue \( \{x, y, z\} \), there are six potential splittings. By Table 1, each entails a different combination of outcomes on the Condorcet triples \( P_{xyz} \) and \( P_{xzy} \). Four of the six combinations directly reveal \( y \succ v z \) or \( z \succ v y \) while two are consistent with every priority ranking of the alternatives \( y \) and \( z \). By varying the alternative \( x \in X \setminus \{y, z\} \), one can resolve the ambiguous cases and determine the revealed priority between any two alternatives \( y \) and \( z \) (see Corollary 1 of the Appendix).

In fact, Table 1 also suggests how to define the amendment rule \( \alpha_v \). Intuitively, \( x \) is revealed to amend a higher priority alternative \( b \) if the Condorcet triples \( P_{xbc} \) and \( P_{xcb} \) for every alternative \( c \in X \setminus \{x, b\} \) with intermediate priority yield type-(a) or type-(c) outcomes. Intuitively, this reveals that \( x \) must appear in the same sub-issue as \( b \) (see Figure 7). Based on this insight, \( \alpha_v(x) \) may then be defined as the highest priority alternative(s) that \( x \) is revealed to amend (see Definition 14 and Lemma 2 of the Appendix).

### 6 Comparison to Euro-Latin and Anglo-American Agendas

In this section, I study the relationship of Euro-Latin and Anglo-American agendas to other priority agendas. I show that four well-known properties of these agendas extend to all priority agendas; and, highlight a natural way to distinguish Euro-Latin and Anglo-American agendas from other priority agendas.

#### 6.1 Four Properties in Common

One implication of Propositions 1-3 is that three well-known properties of Euro-Latin and Anglo-American agendas — Preference Monotonicity, Self-Consistency, and Marginalization — extend to all priority agendas.\(^{33}\)

In this section, I show that the same is true for an appealing monotonicity condition related to priority.

Given a priority agenda \( \text{Pr}_X \) defined by \( (\succcurlyeq, \alpha) \) and an alternative \( x \in \tilde{X}_j \), fix an alternative \( y \in X \) (if any) such that: (i) \( y \in \tilde{X}_{j-1} \); or, (ii) \( y \in X_j \) and \( x \) is amended by every alternative that amends \( y \). Let \( \text{Pr}_X^{\uparrow x} \) denote the priority agenda obtained by permuting the labels of the terminal nodes in \( \text{Pr}_X \) marked by \( x \) and \( y \). The idea is that, relative to \( \text{Pr}_X \), \( \text{Pr}_X^{\uparrow x} \) (weakly) improves the priority of \( x \) by swapping it with an alternative having (i) immediately higher priority or (ii) the same priority but (weakly) fewer amendments.

\(^{32}\)The two outcome pairs with \( v(P_{xyz}, \{x, y, z\}) = y \) reveal \( y \succ v z \); and, the two with \( v(P_{xzy}, \{x, y, z\}) = z \) reveal \( z \succ v y \).

\(^{33}\)Preference Monotonicity and Marginalization follow from Propositions 1 and 3, respectively. Technically, Self-Consistency follows from Theorem 2 (rather than Proposition 2): if a rule \( v \) satisfies IS, ILA, and M, then so does the restricted rule \( v_A \).
Priority Monotonicity For every \( P \in \mathcal{P} \) and \( x \in \mathcal{X} \), \( UNE[Pr_x; P] = x \) only if \( UNE[Pr_{x'}^x; P] = x \).

This property posits a straightforward relationship between the structure of a priority agenda and its voting outcomes. This relationship is well-known to hold for Euro-Latin and Anglo-American agendas (Moulin [1986, 1991]; Jung [1990]). In fact, as the next result shows, it holds for all priority agendas.

**Proposition 4** Every priority agenda satisfies Priority Monotonicity.

### 6.2 A Key Distinction

There is an substantial literature on neutral and Condorcet-consistent mappings, called tournament solutions, which select a subset of the feasible alternatives for every issue \( A \) and tournament \( M_A \) on \( A \). Two of the best known tournament solutions are the Top Cycle and the Banks Set:

**Definition 12** The **Top Cycle** \( TC(M_A) \) is the subset of alternatives in \( A \) that (directly or indirectly) dominate every other alternative according to \( M_A \). Formally, \( TC(M_A) \equiv \{ x \in A : xM_A...M_Aa \text{ for all } a \in A \setminus x \} \).

To define the Banks Set, some additional notation is required. A sequence \( \langle b \rangle = b_1, ..., b_m \) of alternatives in \( A \) is an \( M_A \)-transitive chain if \( b_iM_Ab_j \) for all \( j > i \). In turn, an \( M_A \)-transitive chain is maximal if there exists no alternative \( a \in A \setminus \langle b \rangle \) such that the sequence \( \langle a, b \rangle \equiv a, b_1, ..., b_m \) is an \( M_A \)-transitive chain.

**Definition 13** The **Banks Set** \( BA(M_A) \) is the subset of alternatives in \( A \) that are at the top of some maximal \( M_A \)-transitive chain. Formally, \( BA(M_A) \equiv \{ x \in A : x = b_1 \text{ for some maximal } M_A \text{-transitive chain } \langle b \rangle \} \).

Two early papers show the close connection between these tournament solutions and agendas: Miller [1977] showed that the Top Cycle coincides with the alternatives selected from some Euro-Latin agenda; and, Banks [1985] showed that "his" set coincides with the alternatives selected from some Anglo-American agenda.

To formalize these results, let \( \Sigma(A) \) denote the set of permutations (or bijections) on \( A \); let \( T^\sigma_A \) denote the agenda obtained by permuting the labels of the terminal nodes in \( T_A \) using \( \sigma \in \Sigma(A) \); and, let

\[
V_\Sigma(T_A; P) = \bigcup_{\sigma \in \Sigma(A)} UNE[T^\sigma_A; P]
\]

denote the collection of outcomes selected at profile \( P \) for some re-labeling \( T^\sigma_A \) of the agenda \( T_A \). (For a priority agenda, re-labeling amounts to nothing more than re-prioritizing the alternatives on the agenda.) Then, the foundational results of Miller and Banks establish the following equivalence for each decision problem \((P, A)\):

(i) \( V_\Sigma(EL_A; P) = TC(M^P_A) \) for every Euro-Latin agenda \( EL_A \); and,

(ii) \( V_\Sigma(AA_A; P) = BA(M^P_A) \) for every Anglo-American agenda \( AA_A \).

The next proposition provides a natural extension of their results to the entire class of priority agendas:

---

34 Since the particular profile \( P \) that induces the majority relation is not material, I suppress it in the notation.
Proposition 5 For every decision problem \((P, A) \in P \times X\) and priority agenda \(P_A\) on \(A\):

\[(i')\] \(V_\Sigma(P_A; P) \subseteq TC(M^P_A)\); and,

\[(ii')\] \(V_\Sigma(P_A; P) \supseteq BA(M^P_A)\).

Part (i') follows from McKelvey and Niemi’s [1978] insight that, for every agenda, the sophisticated outcome only depends on the Top Cycle. It is included only as a point of contrast to (i). The real contribution is part (ii'). For a given \(x \in BA(M^P_A)\), the proof defines a sequence \(\langle a \rangle\) on \(A\) (called a universal sequence) such that \(x\) is selected from every priority agenda \(P_A\) when the alternatives are re-prioritized to conform with \(\langle a \rangle\). This extends the work of Banks [1985], who leverages the algorithm of Shepsle and Weingast [1984] to construct a sequence (called a sophisticated sequence) such that \(x \in BA(M^P_A)\) is selected from the associated Anglo-American agenda. In general, there may be more than one sophisticated sequence for a given \(x \in BA(M^P_A)\). The key insight is that one of these sequences must be universal for \(x\) (see Claim 26 of the Appendix).

Combined with observations (i) and (ii), Proposition 5 implies the following relationships:

\[BA(M^P_A) = V_\Sigma(\Lambda A_A; P) \subseteq V_\Sigma(P_A; P) \subseteq V_\Sigma(EL_A; P) = TC(M^P_A).\]  

In other words, Anglo-American and Euro-Latin agendas capture the extremes of what can be implemented by priority agenda; and, all other priority agendas display an intermediate ability to “discriminate” among outcomes.

Formally, \(T_A\) is more discriminating than \(\hat{T}_A\) if \(V_\Sigma(T_A; P) \subseteq V_\Sigma(\hat{T}_A; P)\) for every profile \(P\).

Interestingly, discrimination seems to be negatively related to marginalization. For every issue (consisting of three or more alternatives), Euro-Latin agendas marginalize two alternatives; and, Anglo-American agendas marginalize one. For every other priority agenda, the number of marginal alternatives depends on the issue:

**Theorem 3** A decision rule \(v\) is implementable by a Euro-Latin (resp. Anglo-American) agenda if and only if it satisfies Issue Splitting and the Independence of Losing Alternatives, and it marginalizes two alternatives (resp. one alternative) for every issue \(A \notin X_1 \cup X_2\).

This result characterizes implementation by Euro-Latin and Anglo-American agenda by emphasizing the similarities between the two procedures. In recent work, Apesteguia et al. [2014] give alternate characterizations that instead emphasize the differences (as described at greater length in the Supplementary Appendix).

### 7 Extensions and Applications

In this paper, I characterize sophisticated voting for two broad classes of agendas — classes defined by structural features that are common in legislative settings. For both classes, I establish a close relationship between the structure of the agenda and its sophisticated voting outcomes. While this pushes our understanding of agenda voting well beyond Euro-Latin and Anglo-American agendas, it also suggests some new avenues of inquiry. To conclude, I briefly touch on some potentially interesting extensions and applications.
7.1 Additional Classes of Agendas

For one, the results make it possible to characterize implementation by classes of agendas besides those in Theorems 1-3 — such as knockout agendas (Example 2) and standardized simple agendas (Example 4).

(i) Knockout Agendas: Agendas in this class have the feature that none of the questions leaves alternatives uncontested. Clearly, this feature requires the splitting of each issue to form a partition. To formalize, say that \( v \) is completely splittable for \( A \) if there exists a splitting \((B, C)\) of \( A \) such that \( B \cap C = \emptyset \). Then:

**Complete Issue Splitting** For every issue \( A \notin X_1 \cup X_2 \), \( v \) is completely splittable.

What is more, this property is sufficient to distinguish knockout agendas from other simple agendas:

**Remark 6** A decision rule \( v \) is implementable by a knockout agenda if and only if it satisfies Complete Issue Splitting and the Independence of Losing Alternatives.

The main interest of this characterization lies in its connection to Theorem 1.\(^{35}\) It shows that simple agendas reflect a natural way to extend knockout agendas to include questions with uncontested alternatives.

(ii) Standard Agendas: For agendas in this class (see Definition 8), the status quo alternative \( x^\ast \) must appear in every terminal sub-game. By a well-known folk result (see e.g., Lemma 2.2 of Iglesias et al. [2014]), it follows that \( x^\ast \) must be marginal on \( X \).\(^{36}\) So, implementation by a standard agenda imposes the Weak Marginalization requirement that \( v \) has a marginal alternative on \( X \). However, this condition is not sufficient to distinguish standardized agendas from other simple agendas. Among non-standard agendas, it is also satisfied by Euro-Latin agendas, for instance. To exclude this type of agenda, one could rule out issues where \( v \) marginalizes \( x^\ast \) (the marginal alternative on \( X \)) together with a second alternative.

**Weak* Marginalization** There exists an alternative \( x^\ast \in X \) such that:

- \( v \) marginalizes \( x^\ast \) on the (universal) issue \( X \); and,
- \( v \) does not marginalize multiple alternatives on any issue \( A \notin X_1 \cup X_2 \) containing \( x^\ast \).

As it turns out, this property is sufficient to distinguish standardized agendas from all other simple agendas.

**Remark 7** (i) Every decision rule \( v \) implementable by a standard agenda satisfies Weak Marginalization; and, (ii) a decision rule \( v \) is implementable by a standardized simple agenda if (and only if) it satisfies Issue Splitting, the Independence of Losing Alternatives, and Weak* Marginalization.

\(^{35}\)Alternatively, one could characterize knockout agendas with Division Consistency (see footnote 21 above): \( v \) is implementable by a knockout agenda if and only if it satisfies Division Consistency and selects the Condorcet winner whenever it exists.

\(^{36}\)In particular, fix a profile \( P \) where \( x^\ast \) is not the Condorcet winner. In the first step of the “backward induction” algorithm, an alternative in \( X \setminus x^\ast \) can survive if and only if it is majority preferred to \( x^\ast \). In each subsequent step of the algorithm, it follows that some alternative which is majority preferred to \( x^\ast \) must survive. As a result, \( x^\ast \) cannot be the outcome for \((P, X)\).
Part (i) of this remark is a direct consequence of a more general observation about the relationship between the decision rule \( v : P \times (X \setminus x^*) \to X \setminus x^* \) implemented by an agenda \( T \) on \( X \setminus x^* \) and the rule \( v^* : P \times X \to X \) implemented by the standardized agenda \( T^* \) on \( X \). Where \( \hat{M}_A^P(x^*) \equiv \{ a \in A : aM_A^x x^* \} \) denotes the set of alternatives that are majority-preferred to \( x^* \) for each \((P, A) \in P \times X\), it is not difficult to show that

\[
v^*(P, A) = \begin{cases} 
v(P, A) & \text{if } x^* \notin A \\
x^* & \text{if } x^* \text{ is the Condorcet winner on } (P, A) \\
v(P, \hat{M}_A^P(x^*)) & \text{otherwise}. \end{cases}
\] 

This formula clearly shows how standardization provides a systematic advantage to the alternatives that are majority-preferred to \( x^* \). If the status quo \( x^* \) is not the Condorcet winner on \((P, A)\), then the standardized agenda selects \( v(P, \hat{M}_A^P(x^*)) \) when the original agenda would have selected \( v(P, A \setminus x^*) \).

### 7.2 Agenda Manipulation

In addition, the results help clarify the scope of manipulation that can arise once the agenda is set.

(i) **Preference Monotonicity:** This property imposes some natural limitations on agenda manipulation. For one, it discourages voters from misrepresenting their preferences by downgrading an alternative. As noted by Altman et al. [2009], it also discourages a natural kind of manipulation by the alternatives — which can be viewed as candidates in an election. In particular, it prevents them from gaining an advantage by “throwing” majority comparisons with other candidates. Famously, this type of manipulation occurred during the 2012 Olympic badminton tournament (see Pauly [2013]).

Given its appealing incentive effects, it would be worth identifying structural agenda features that are necessary and sufficient for Preference Monotonicity. Proposition 1 goes some way towards this goal by identifying two agenda features that are sufficient (but not necessary). Previously, the only agendas known to exhibit this property were knockout agendas and Anglo-American agendas (as noted in Section 4.2 above). Having said this, the gap between sufficiency and necessity may be significant. As the following remark shows, some agendas that look quite different from simple agendas nonetheless satisfy Preference Monotonicity.

**Remark 8** There exist discontinuous (and non-repetitive) agendas that satisfy Preference Monotonicity.

Ultimately, this suggests that it might be difficult to identify features that are both necessary and sufficient.

(ii) **Strategic Candidacy:** Besides “throwing” pairwise comparisons, Dutta et al. [2001] suggest another way for a candidate to manipulate the outcome — namely, by “dropping out” of the election.

To formalize, let \( A = \{ a_1, ..., a_n \} \) denote the set of candidates; and, suppose that, in every candidate preference profile \( P_c \equiv (\succ_{a_1}, ..., \succ_{a_n}) \in P_c \), each \( a_i \in A \) ranks herself first (i.e., \( a_i \succ_{a_i} x \) for all \( x \in A \setminus a_i \)).

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37By extending this approach, one can also give a recursive formula to determine the outcome on any priority agenda.
Then, a triple \( (T_A; P, P^c) \) defines a two-stage game where: (1) a subset of the candidates \( A^* \subseteq A \) independently decides to enter the election; and then, (2) the voters then select one of the entering candidates using the pruned agenda \( T_{A|A^*} \). Let \( EQ[T_A; P, P^c] \) denote the equilibrium outcomes (if any) of this game where:

- the outcome of (2) is \( UNE[T_{A|A^*}; P] \) given the set of candidates \( A^* \) who decide to enter in (1); and,
- \( A^* \) is a pure Nash equilibrium outcome of (1) induced by anticipating \( UNE[T_A|B; P] \) for all \( B \subseteq A \).

Dutta et al. [2002] consider the case where the agenda is Anglo-American; and, characterize the set of outcomes that results when the candidate preferences and priority are allowed to vary. Given a pair \( (T_A, P) \), let

\[
\tilde{V}_\Sigma(T_A; P) \equiv \bigcup_{\sigma \in \Sigma(A)} EQ[T_{\sigma A}; P, P^c]
\]

denote the outcomes selected for some re-labeling \( \sigma \in \Sigma(A) \) and some candidate preferences \( P^c \in P^c \). Dutta et al. show that \( \tilde{V}_\Sigma(\text{AA}_A; P) \) defines a tournament solution called the Candidate Stable Set.\(^{38}\)

Using Proposition 5, one can generalize their result to priority agendas. Where \( CS(M_A) \) denotes the Candidate Stable Set associated with \( M_A \), one obtains a natural analog of the relationships in (†) above:

**Remark 9** For every decision problem \( (P, A) \in P \times X \) and priority agenda \( Pr_A \) on \( A \):

\[
CS(M^p_A) = \tilde{V}_\Sigma(AA_A; P) \subseteq \tilde{V}_\Sigma(Pr_A; P) \subseteq \tilde{V}_\Sigma(EL_A; P) = TC(M^p_A).
\]

This remark suggests that the results of the current paper might be used to pursue the study of strategic candidacy even further. In that regard, the comments in Section 7.3(i) below apply equally well here: it would be helpful to understand more about how \( \tilde{V}_\Sigma(Pr_A; \cdot) \) depends on the amendment structure of the agenda \( Pr_A \); and, how \( \tilde{V}_\Sigma(T_A; \cdot) \) is determined in the more general case where \( T_A \) is not a priority agenda.

### 7.3 Strategic Agenda-Setting

Finally, the results shed light on the possibility of strategic behavior at the earlier stage of agenda-setting.

**(i) Discriminability:** In the literature, the (size of the) set \( V_\Sigma(T_A; \cdot) \) is generally interpreted as a measure of how much control an “all-powerful” chairman can exert by re-arranging the alternatives on the agenda. On this basis, more discriminating agendas (like Anglo-American agendas) are generally viewed more favorably than less discriminating agendas (like Euro-Latin agendas).

Taken together, Theorem 3 and Proposition 5 suggest that the ability of a priority agenda to discriminate among outcomes is closely tied to its amendment structure. One idea is that a priority agenda becomes more discriminating as it becomes “bushier” (i.e., as \( \alpha(x) \) increases in terms of priority \( \succeq \) for a given alternative

\(^{38}\)See Definition 16 of the Supplementary Appendix. This solution is nested between the Banks Set and the Uncovered Set (Miller [1980]) — another tournament solution which specializes the Top Cycle (see Definition 17 of the Supplementary Appendix).
For a fixed universe of alternatives $X$, 
\textit{bushiness} defines a partial ordering of the priority agendas on $X$. However, this partial ordering does not coincide with the partial ordering defined by discriminability:

\textbf{Remark 10} For $|X| = 4$ alternatives, the discriminability and bushiness orderings of priority agendas disagree.

Ideally, one would hope to have a better understanding of the connection between structure and discriminability for priority agendas.\textsuperscript{39} Of particular interest would be to determine the sub-class of priority agendas that always selects from the Uncovered Set (see footnote 38 above), the largest tournament solution between the Banks Set and the Top Cycle whose elements are Pareto optimal. Dutta et al.’s [2004] results on strategic agenda-setting with Anglo-American agendas extend directly to this class of agendas; and, it would be worth investigating whether the same is also true for Duggan’s [2006] results.

\textbf{(ii) Self-Consistency:} A different (but equally plausible) way to manipulate decision rules is by adding “dummy” alternatives to the agenda (Miller [1995]). If the class of “permissible” agendas satisfies Self-Consistency, then this kind of manipulation cannot provide any additional benefit: for every decision rule that can be obtained in this way, there is a way to achieve the same thing by restructuring the original agenda.

Given this observation, it would be worth identifying some broad classes of agendas that satisfy Self-Consistency. Proposition 1 makes some progress — by showing that the class of simple agendas $S$ exhibit this property while the class of recursive agendas $R$ do not. Example 7 goes even further — by showing that there exist classes of agendas which are nested (strictly) between $S$ and $R$ that satisfy Self-Consistency. In fact, there exists a maximal sub-class $R^{sc} \subseteq R$ with this property:

\textbf{Remark 11} For every class of agendas $T$, there exists a maximal sub-class which satisfies Self-Consistency.

Along the lines of the current paper, one might endeavor to determine the structural features that define the agendas in $R^{sc}$; and, to characterize the family of decision rules that are implemented by these agendas.

\section{References}


\textsuperscript{39}Indeed, one might even hope to identify such a connection for the broader classes of simple agendas or recursive agendas (see Tables 3 and 4 of the Supplementary Appendix).


9 Mathematical Appendix

9.1 Primer on Graph Theory

A graph \( G = (Q, E) \) is a pair consisting of: a collection of nodes \( Q \); and, a collection of (undirected) edges \( E \), each of which connects two nodes \( q \) and \( \tilde{q} \) in \( Q \). It is labeled if each node \( q \in Q \) has a label \( \ell(q) \); finite if \( |Q| \) is finite; and, loopless if there exist no edges in \( E \) (called loops) that connect a node in \( Q \) to itself. A (simple) path in \( G \) is a non-repeating sequence of nodes in \( Q \) such that each adjacent pair in the sequence is connected by an edge in \( E \).

A tree \( T = (Q, E) \) is a loopless graph such that every pair of distinct nodes in \( Q \) is connected by a unique simple path. In a tree, nodes can be ordered in a natural way. In particular, a node \( q' \in Q \) is between two nodes \( q, q'' \in Q \) if the path from \( q \) to \( q'' \) (inclusive of the end nodes) passes through \( q' \).

A tree is rooted if one node \( r \in Q \) is specially designated as the root. In a rooted tree \( T_r = (Q, E, r) \), betweenness defines a partial ordering \( \leq_E \) on the nodes where \( q \leq_E \tilde{q} \) for \( q, \tilde{q} \in Q \) if \( q \) is between the root \( r \) and the node \( \tilde{q} \). (It is easy to see that \( \leq_E \) is reflexive, transitive, and antisymmetric.) Equivalently, the nodes exhibit a strict partial ordering \( <_E \) defined by \( q <_E \tilde{q} \) for \( q, \tilde{q} \in Q \) if \( q \leq_E \tilde{q} \) and \( q \neq \tilde{q} \). Alternatively, a rooted tree can be defined directly in terms of the nodes \( Q \) and a partial ordering \( \preceq \) (or a strict partial ordering \( < \)) on \( Q \). In either case, the collection of edges \( E_c \) is defined by \( \{q, \tilde{q}\} \in E_c \) if \( q <_E \tilde{q} \) and \( q < q' <_E \tilde{q} \) for no \( \tilde{q} \in Q \). In a rooted tree \( T_r \), \( \tilde{q} \in Q \) is a predecessor (resp. successor) of \( q \in Q \) if \( \{q, \tilde{q}\} \in E_c \) and \( \tilde{q} < q \) (resp. \( q < \tilde{q} \)). In turn, a node \( q \in Q \) is said to be terminal if it has no successors; and, non-terminal otherwise. By construction, every non-root node of a rooted tree has a unique predecessor; and, every non-terminal node has a successor.

Finally, a binary tree is a rooted and labeled tree such that every non-terminal node of \( Q \) has exactly two successors.

9.2 Proof of Remark 1

Suppose that \( v \) is implemented by \( T_X \). Then, \( v(P^A, X) \equiv UNE[T_X; P^A] \) and \( v(P, A) \equiv UNE[T_X[A]; P] \) by definition. To see that \( UNE[T_X; P^A] = UNE[T_X[A]; P^A] \), note that "backward induction" determines the UNE on any agenda. In any terminal subgame, it selects the Condorcet winner. One can then delete the Condorcet loser and repeat the argument on the resulting (smaller) agenda. Since \( P^A \) and \( P \) coincide on \( A \), \( UNE[T_X[A]; P^A] = UNE[T_X[A]; P] \) as well. Combining these equalities gives \( v(P^A, X) = UNE[T_X; P^A] = UNE[T_X[A]; P^A] = UNE[T_X[A]; P] = v(P, A) \) as required.

9.3 Proof of Remark 2

Consider the standardized agenda \( T^*_A \) obtained by appending \( x^* \) at each terminal node of the base agenda \( T_A \). (\( \Rightarrow \)) History-independence: Fix two non-terminal nodes \( q \) and \( q' \) of \( T^*_A \) labeled by \( D \subseteq A \). If \( |D| = 2 \), then \( D = \{x^*, d\} \) so that \( \{\ell(q_1), \ell(q_2)\} = \{\ell(q'_1), \ell(q'_2)\} \) by definition of \( T^*_A \). Otherwise, \( x^* \in u(q), u(q') \) by definition of \( T^*_A \); and, \( \{\ell(q_1), \ell(q_2)\} = \{\ell(q'_1), \ell(q'_2)\} \) follows directly from the history-independence of \( T_A \). Persistence: Fix a non-terminal node \( q \) of \( T^*_A \). If \( |\ell(q)| = 2 \), then \( T^*_A \) is (trivially) persistent at \( q \) because no alternative is uncontested. Otherwise, \( x^* \in u(q) \) by definition of \( T^*_A \); and, persistence at \( q \) follows directly from the persistence of \( T_A \). (\( \Leftarrow \)) History-independence and persistence of \( T_A \) follow directly from the fact that these properties hold for \( T^*_A \).

9.4 Proof of Remark 3

Fix a non-terminal node \( q \) of \( T_X \). (\( \Rightarrow \)) Since \( T_X \) is non-repetitive, \( c_i(q) \neq \emptyset \). Since \( T_X \) is continuous, some \( x_i \in c_i(q) \) labels exactly one terminal node below \( q \), for \( i = 1, 2 \). (\( \Leftarrow \)) By assumption, \( c_i(q) \neq \emptyset \) for \( i = 1, 2 \). So, \( T_X \) is non-repetitive.
By assumption, some \( x_i \in c_1(q) \) labels exactly one terminal node below \( q \), for \( i = 1, 2 \). So, \( T_X \) is continuous.

### 9.5 Proof of Remark 4

**Claim 1** *Every history-independent agenda is non-repetitive.*

**Proof.** (i) Fix a history-independent agenda \( T_X \); and, consider a non-terminal node \( q \) where \( \ell(q) \equiv B \). By way of contradiction, suppose \( \ell(q_1) = B \) and \( \ell(q_2) \equiv C \subseteq B \). By history-independence, the successors of \( q_1 \) are labeled \( B \) and \( C \). Repeating this argument, there is an infinite path down the agenda from \( q \) where every node is labeled by \( B \). But, this contradicts the fact that \( T_X \) is finite. ■

**Claim 2** *Every simple agenda is continuous.*

**Proof.** Fix a simple agenda \( S_X \). The proof is by strong induction on \( |X| = m \).

The base cases \( m = 2, 3 \) follow from Claim 1. To complete the induction, suppose the claim holds for \( m \leq n \) and consider the case \( m = n + 1 \). Let \( r \) denote the root; and, \( r_1 \) and \( r_2 \) its two successors. Since \( S_X \) is simple, the agenda \( S'_X \) starting from \( r_i \) is simple for \( i = 1, 2 \). Since \( S_X \) is non-repetitive by Claim 1, \( \ell(r_i) \subseteq X \) for \( i = 1, 2 \). So, each \( S'_X \) is continuous by the induction hypothesis. By Claim 1, each \( S'_X \) is also non-repetitive. Let \( r_{i1} \) and \( r_{i2} \) denote the successors of \( r_i \). By Remark 3, some \( x_{i2} \in c_2(r_i) \) labels exactly one terminal node below \( r_{ij} \) for \( j = 1, 2 \).

Given Remark 3, the proof is complete if \( x_{i1} \in c_1(r_i) \) or \( x_{i2} \in c_1(r_i) \) for \( i = 1, 2 \). By way of contradiction, suppose \( x_{i1}, x_{i2} \in u(r) \). Now, consider any terminal node \( t \) below \( r_i \). By persistence, there exists a node \( q^t \) (between \( r \) and \( t \)) with a successor labeled \( u(r) \). Since \( x_{i1} \in c_1(r_i) \) and \( x_{i2} \in c_2(r_i) \), it cannot be any node between \( r_i \) and \( t \). So, the only possibility is that \( q^t = r \). Hence, \( u(r) \in \{ \ell(r_1), \ell(r_2) \} \). Without loss of generality, suppose \( \ell(r_1) = u(r) \). Then, it must be that \( \ell(r_2) = c(r) \cup u(r) = \ell(r) \). But, this contradicts the fact that \( S_X \) is non-repetitive (by Claim 1). ■

**Proof of Remark 4.** (i) By Claims 1 and 2. (ii) It is easy to check that the left-hand agenda in Figure 5 is recursive. However, it is not simple. Since the two nodes labeled \( \{x, b, c\} \) give rise to distinct sub-agendas, it violates history-independence. (Still, it is persistent: the root \( r \) is the only node with uncontested alternatives; and, both of its successors have successors labeled \( u(r) = \{x, b, c\} \). ■

**Claim 3** *Given a simple agenda \( S_X \) with a non-terminal node \( q \) such that \( u(q) \neq \emptyset \) and a terminal node \( t \) below \( q \), there exists a unique non-terminal node \( q^t \) between \( q \) and \( t \) such that \( u(q) \in \{ \ell(q^t_1), \ell(q^t_2) \} \). Consequently:

(a) \( \emptyset \subseteq \ell(q^t_1) \cap c(q) \subseteq \ell(q^t_1) \cap (q^t) \) and \( u(q) = \ell(q^t_2) \subseteq \ell(q^t) \) at node \( q^t \); and,

(b) \( u(q) \subseteq \ell(q^t_1) \cap (q) \) for \( i = 1, 2 \) at each node \( q \) strictly between \( q \) and \( q^t \).

What is more, the contested outcomes at the nodes between \( q \) and \( q^t \) satisfy the following:

(a) \( \emptyset \subseteq \ell(q^t_1) \cap c(q) \subseteq \ell(q^t_1) \cap (q^t) \) at node \( q^t \); and,

(b) \( \emptyset \subseteq c(q) \subseteq \ell(q^t) \cap c(q) \) at each node \( q \) strictly between \( q \) and \( q^t \).

**Proof.** The fact that \( S_X \) is persistent and non-repetitive (by Claim 1) implies the first part of the claim. (a.i) Without loss of generality, \( u(q) = \ell(q^t_1) \) by the first part of the claim. So, \( \ell(q^t_1) \supseteq \ell(q^t_1) \cap \ell(q^t_2) = \ell(q^t) \backslash u(q) = \ell(q^t) \cap c(q) \supseteq \emptyset \) as required. And, since \( S_X \) is non-repetitive, \( \ell(q^t) \cap c(q) \) for \( i = 1, 2 \). (a.ii) For the lower bounds: \( u(q) \subseteq \ell(q^t_1) \) by the first part of the claim. For the upper bounds: \( \ell(q^t) \subseteq \ell(q^t_1) \) since \( S_X \) is non-repetitive. (b.i) For the lower bound: note that \( c_1(q^t) \supseteq \ell(q^t) \cap c(q) \) by (a.i); and, \( c_2(q^t) \supseteq \emptyset \) since \( S_X \) is non-repetitive. For the upper bound: note that \( c(q^t) = \ell(q^t) \cap c(q) \) if \( \ell(q^t) = \ell(q^t_1) \cap c(q) \). (b.ii) For the lower bound: note that \( c(q^t_2) \supseteq \emptyset \) since \( S_X \) is non-repetitive. For the upper bound: note that \( c(q^t_2) = \ell(q^t) \cap c(q) \) if \( \tilde{c_1}, \tilde{c_2} \) partitions \( \ell(q^t) \cap c(q) \) and \( \ell(q^t_2) = u(q) \cup \tilde{c} \) for \( i = 1, 2 \). ■
9.6 Proof of Theorem 1

Subsection 1 establishes sufficiency, while subsections 2 and 3 establish necessity and uniqueness, respectively.

9.6.1 Proof of Sufficiency

Remark 12 For \( A = B \cup C \), the following are equivalent: (i) \( B, C \neq A \); (ii) \( B \cap C \neq B, C \); and, (iii) \( B \setminus C, C \setminus B \neq \emptyset \).

Proof. (i) \( \Rightarrow \) (ii): Suppose \( B \cap C = B \). Then, \( C = (B \cap C) \cup C = B \cup C = A \) which is a contradiction. (ii) \( \Rightarrow \) (iii): Suppose \( B \setminus C = \emptyset \). Then, \( B = (B \cap C) \cup (B \setminus C) = (B \cap C) \cup \emptyset = B \cap C \) which is a contradiction. (iii) \( \Rightarrow \) (i): Suppose \( B = A \). Then, \( C \setminus B = C \setminus A = \emptyset \) which is a contradiction. ■

Claim 4 If \( v \) satisfies IS and ILA, then \( v(P, A) = v(P', A) \) for all \( P, P' \in P \) that coincide on \( A \in X \).

Proof. The proof is by induction on \( |A| \). The base case \( |A| = 2 \) follows from ILA. To complete the induction, suppose the claim holds for \( |A| = n \) and consider the case \( |A| = n + 1 \). Then, IS, the induction hypothesis, and the base case imply \( v(P, A) = v(P, \{v(P, B), v(P, C)\}) = v(P, \{v(P', B), v(P', C)\}) = v(P', \{v(P', B), v(P', C)\}) = v(P', A) \). ■

Claim 5 If \( v \) satisfies IS and ILA, then \( v(P, A) = v(P^A, X) \) for all \( (P, A) \in P \times X \).

Proof. By ILA, \( v(P^A, X) = \ldots = v(P^A, A) \). Since \( v(P^A, A) = v(P, A) \) by Claim 4, \( v(P^A, X) = v(P, A) \). ■

Claim 6 Suppose \( v \) satisfies IS and ILA. If \( (B, C) \) splits \( A \notin X_1 \cup X_2 \) and \( D \subset A \), then:

(i) \( v(P, D) = v(P, \{v(P, B \cap D), v(P, C \cap D)\}) \); and, (ii) \( (B \cap D, C \cap D) \) splits \( D \) if \( D \neq B \cap D, C \cap D \).

Proof. Fix some \( x \in A \) and let \( P_x \) coincide with \( P \) except that \( x \) is demoted to Condorcet loser on \( A \). Then, \( v(P_x, A \setminus x) = v(P_x, A \setminus x) = v(P_x, A) = v(P_x, \{v(P_x, B), v(P_x, C)\}) = \ldots = v(P, \{v(P, B \setminus x), v(P, C \setminus x)\}) \) by Claim 4, ILA, and IS. Part (i) follows by repeated application of this reasoning. For part (ii), observe that \( D \neq B \cap D, C \cap D \) implies \( B \cap C \neq D \neq B \cap D, C \cap D \) by Remark 12. So, \( (B \cap D, C \cap D) \) splits \( D \) by part (i). ■

Claim 7 Suppose \( v \) satisfies IS and ILA. If \( (B, C) \) splits \( A \notin X_1 \cup X_2 \) and \( \emptyset \subset B \cap C \subseteq D \subseteq C \), then either:

(i) \( (B', B \cap C) \) splits \( D \) for some \( B' \subset D \); or, (ii) \( (B', C') \) splits \( D \) for some \( B', C' \supseteq B \cap C \).

Proof. Let \( (B', C') \) denote a splitting of \( D \). To establish the result, it suffices to show: (1) \( B \cap C \subseteq B' \) or \( B \cap C \subseteq C' \); and, (2) \( B \cap C = C' \) if \( B \cap C \subseteq B' \). Given (1) and (2), the only two possibilities for \( (B', C') \) are (i) and (ii).

(1) By way of contradiction, suppose \( B \cap C \not\subseteq B', C' \). Fix alternatives \( b \in (B \cap C) \setminus C' \) and \( c \in (B \cap C) \setminus B' \). Since \( B \cap C \subseteq D \), there exists some \( d \in D \setminus (B \cap C) \). Without loss of generality: (a) \( d \in B' \cap C' \); or, (b) \( d \in B' \setminus C' \) (since \( d \in C' \setminus B' \) is similar). Since \( (B, C) \) splits \( A \), \( v(P, \{b, c, d\}) = v(P, \{v(P, \{b, c\}), v(P, \{b, c, d\})\}) \) by Claim 6(i).

For cases (a) and (b), I show that this entails a contradiction. (a) Since \( (B', C') \) splits \( D \),

\[
v(P, \{b, c, d\}) = v(P, \{v(P, \{b, d\}), v(P, \{c, d\})\}).
\]

So, \( v(P_{bcd}, \{b, c, d\}) = c \) for the Condorcet triple \( P_{bcd} \). Combining this with the formula in the last paragraph implies

\[
c = v(P_{bcd}, \{b, c, d\}) = v(P_{bcd}, \{v(P_{bcd}, \{b, c\}), v(P_{bcd}, \{b, c, d\})\}) = v(P_{bcd}, \{b, c\}) = b.
\]
which is a contradiction. (b) Since \((B', C')\) splits \(D\), \(v(P, \{b, c, d\}) = v(P, \{v(P, \{b, d\}), c\})\). So, \(v(P_{bcd}, \{b, c, d\}) = c\). Then, by the same reasoning as in case (a), this entails the contradiction that \(c = v(P_{bcd}, \{b, c, d\}) = b\).

(2) Since \(B \cap C \not\subseteq B', B \cap C \subseteq C'\) by (1). By way of contradiction, suppose \(B \cap C \subset C'\). Fix alternatives \(b \in B' \setminus C'\), \(c \in (B \cap C) \setminus B'\), and \(d \in C' \setminus (B \cap C)\). Since \(B \cap C \subset C'\), \(b \in B' \setminus (B \cap C)\). There are two cases: (a) \(d \in B \cap C\); and, (b) \(d \in C' \setminus B'\). Since \(B'\) splits \(A\), \(v(P, \{b, c, d\}) = v(P, \{c, v(P, \{b, c, d\})\})\) by Claim 6(i).

For cases (a) and (b), I show that this entails a contradiction. (a) Since \((B', C')\) splits \(D\),

\[
v(P, \{b, c, d\}) = v(P, \{v(P, \{b, d\}), v(P, \{c, d\})\}).
\]

So, \(v(P_{bcd}, \{b, c, d\}) = b\) for the Condorcet triple \(P_{bcd}\). Combining this with the formula in the last paragraph implies

\[
b = v(P_{bcd}, \{b, c, d\}) = v(P_{bcd}, \{c, v(P_{bcd}, \{b, c, d\})\}) = v(P_{bcd}, \{b, c\}) = c,
\]

which is a contradiction. (b) Since \((B', C')\) splits \(D\), \(v(P, \{b, c, d\}) = v(P, \{b, v(P, \{c, d\})\})\). So, \(v(P_{bcd}, \{b, c, d\}) = c\). Then, by the same reasoning as in case (a), this entails the contradiction that \(b = v(P_{bcd}, \{b, c, d\}) = c\).

\[\text{Claim 8}\]

If \(v\) satisfies IS and ILA, then there exists a unique splitting for all \(A \not\in X_1 \cup X_2\).

\[\text{Proof.}\] The proof is by induction on \(|A|\). For the base case \(|A| = 3\), Table 1 shows that every potential splitting of \(A\) yields a distinct pair of outcomes on the two Condorcet triples. So, the splitting of \(A\) must be unique.

To complete the induction, suppose the claim holds for \(|A| = n\) and consider \(|A| = n + 1\). By way of contradiction, suppose \((B, C)\) and \((B', C')\) are distinct splittings of \(A\). First, suppose \((B, C)\) and \((B', C')\) partition \(A\). Then, there exists some \(x \in A\) such that \((B \setminus x, C \setminus x)\) and \((B' \setminus x, C' \setminus x)\) are distinct partitions of \(A \setminus x\), which contradicts the induction hypothesis. Next, suppose \(x \in B \cap C\) for some \(x \in A\). If \(x \in B' \cap C'\), then \((B \setminus x, C \setminus x)\) and \((B' \setminus x, C' \setminus x)\) are distinct splittings of \(A \setminus x\), which again contradicts the induction hypothesis. Finally, suppose \(x \in B' \setminus C'\). There are two cases for \((B \setminus x, C \setminus x)\) and \((B' \setminus x, C' \setminus x)\): either (i) they coincide; or, (ii) they are distinct.

(i) Without loss of generality, \(B \setminus x = B' \setminus x\) and \(C \setminus x = C'\) (the only other possibility is symmetric). Since \((B, C)\) splits \(A\), there exist alternatives \(b \in B \setminus C\) and \(c \in C \setminus B\). Since \(B' \setminus x = B' \setminus x\) and \(C' \setminus x = C'\), \(b \in B' \setminus C'\) and \(c \in C' \setminus B'\). Now, consider \(D = \{b, c, x\}\). From the observations above: \((B \setminus x, C \setminus x) = \{(x, b), (x, c)\}\) and \((B \setminus D, C' \setminus D) = \{(x, b), (x, c)\}\). For the Condorcet triple \(P_{bxc}\), \(v(P_{bxc}, \{v(P_{bxc}, B \setminus D), v(P_{bxc}, C \setminus D)\}) = v(P_{bxc}, \{x, b\}) = b\) and, \(v(P_{bxc}, \{v(P_{bxc}, B' \setminus D), v(P_{bxc}, C' \setminus D)\}) = v(P_{bxc}, \{b, c\}) = c\). But, this contradicts Claim 6(i).

(ii) First, suppose \(B' \setminus C' \neq x\). Then \((B \setminus x, C \setminus x)\) and \((B' \setminus x, C' \setminus x)\) are distinct splittings of \(A \setminus x\), which contradicts the induction hypothesis. Next, suppose \(B' \setminus C' = x\). Since \((B, C)\) splits \(A\), there exist \(b \in B \setminus C\) and \(c \in C \setminus B\). Now, consider \(D = \{b, c, x\}\). From the observations above: \((B \setminus x, C \setminus x) = \{(x, b), (x, c)\}, x \in B' \setminus C',\) and \(b, c \in C'\). The last two sets include three possibilities: (a) \((B' \setminus D, C' \setminus D) = \{(x, b), (x, c)\}\); (b) \((B' \setminus D, C' \setminus D) = \{(x, b), (x, c)\}\); and, (c) \((B' \setminus D, C' \setminus D) = \{(x, b), (x, c)\}\). For (a)-(b), \(v(P_{bxc}, \{v(P_{bxc}, B \setminus D), v(P_{bxc}, C \setminus D)\}) = v(P_{bxc}, \{x, b\}) = b\) and, \(v(P_{bxc}, \{v(P_{bxc}, B' \setminus D), v(P_{bxc}, C' \setminus D)\}) = v(P_{bxc}, \{b, c\}) = c\). But, this contradicts Claim 6(i). For (c), \(v(P_{bxc}, D) = c\) by Claim 6(ii); and, \(v(P_{bxc}, D) = v(P_{bxc}, \{v(P_{bxc}, B' \setminus D), v(P_{bxc}, C' \setminus D)\}) = v(P_{bxc}, \{v(P_{bxc}, D), v(P_{bxc}, \{b, c\})\}) = v(P_{bxc}, \{b, c\}) = b\). But, this is a contradiction.

\[\text{Note:}\] In the proof of Claims 9 and 10 below, I abuse notation by referring to a node \(q\) of \(S_X^\ell\) by its label \(\ell(q)\). By virtue of Claim 8 however, this abuse does not create any possibility for confusion.

\[\text{Claim 9}\]

If \(v\) satisfies IS and ILA, then \(S_X^\ell\) defines a simple agenda.
Since \( \text{quires} \) m cases

In this section, I frequently abuse notation by referring to a node \( X \). This follows directly from Claims 9-10 above.

Claim 10

If \( v \) satisfies IS and ILA, then \( S_X^v \) implements \( v \).

Proof. By Claim 9, \( S_X^v \) defines an agenda. The proof that it implements \( v \) is by strong induction on \( |X| = m \). The base cases \( m = 2, 3 \) follow from IS and the definition of \( S_X^v \). To complete the induction, suppose the claim holds for \( m \leq n \) and consider the case \( m = n + 1 \).

Let \( B_1 \) and \( C_1 \) denote the successors of the root \( X \) in \( S_X^v \). Consider the restricted decision rules \( v_{B_1} \) and \( v_{C_1} \) on \( B_1 \) and \( C_1 \) (see Definition 10). Note that \( v_{B_1} \) and \( v_{C_1} \) inherit IS and ILA from \( v \); and, \( |B_1|, |C_1| \leq n \) (since \( B_1, C_1 \subset X \) by IS). By the induction hypothesis, it follows that \( S_{B_1}^{v_{B_1}} \) and \( S_{C_1}^{v_{C_1}} \) are persistent. Since \( S_X^v(B_1) = S_{B_1}^{v_{B_1}} \) and \( S_X^v(C_1) = S_{C_1}^{v_{C_1}} \) by construction, this establishes that \( S_X^v \) is persistent at every non-terminal node except possibly the root \( X \).

To check persistence at \( X \), suppose \( B_1 \cap C_1 \neq \emptyset \) and consider any terminal node \( x \) below \( C_1 \). Let \( X, C_1, \ldots, C_n, x \) denote the path from \( X \) to \( x \). First, consider the node \( C_1 \). By construction of \( S_X^v \), (\( B_2, C_2 \) splits \( C_1 \cap B_1 \cap C_1 \). By Claim 7, there are three possibilities: \( B_2 = B_1 \cap C_1 \); \( C_2 = B_1 \cap C_1 \); or, \( B_2, C_2 \supset B_1 \cap C_1 \). In the first two cases, \( C_1 \) is a node between \( X \) and \( x \) with a successor \( B_3 \cap C_3 \). In the third case, consider the node \( C_2 \). Since \( B_1 \cap C_1 \subset C_2 \subset C_1 \), Claim 7 applies. So, one can repeat the argument given for \( C_1 \). Since \( S_X^v \) is finite, continuing inductively in the same vein establishes that one of the nodes \( C_i \) between \( X \) and \( x \) must have a successor \( B_1 \cap C_1 \).

By the same argument, the same is true for any terminal node \( x \) below \( B_1 \). So, \( S_X^v \) is persistent at \( X \).

Claim 10 If \( v \) satisfies IS and ILA, then \( S_X^v \) implements \( v \).

Proof. By Claim 9, \( S_X^v \) defines an agenda. The proof that it implements \( v \) is by strong induction on \( |X| = m \). The base cases \( m = 2, 3 \) follow from IS and the definition of \( S_X^v \). To complete the induction, suppose the claim holds for \( m \leq n \) and consider the case \( m = n + 1 \).

To show that \( v(P, X) = UNE[S_X^v; P] \) for all \( P \in \mathcal{P} \). (To see this, fix some \( (P, A) \in \mathcal{P} \times X \). Since \( v(P, A) = v(P^A, X) \) (by Claim 5) and \( UNE[S_X^v; P^A] = UNE[S_X^v|A; P] \) (by Remark 1), it follows that \( v(P, A) = v(P^A, X) = UNE[S_X^v; P^A] = UNE[S_X^v|A; P] \). In other words, \( S_X^v \) implements \( v \).

9.6.2 Proof of Necessity

Note: In this section, I frequently abuse notation by referring to a node \( q \) of \( S_X \) by its label \( \ell(q) \). Since \( S_X \) is simple (and, thus, history-independent by the discussion in the text), this abuse does not create any possibility for confusion.

Claim 11 If \( S_X \) is a simple agenda where \( B \) and \( C \) are the successors of the root \( X \), then:

\[
UNE[S_X; P] = x \in B \cap C \text{ implies } UNE[S_X(B); P] = x = UNE[S_X(C); P].
\]

Proof. The proof is by strong induction on \( |X| = m \). The base cases \( m = 2, 3 \) follow from the fact that \( S_X \) is non-repetitive (by Claim 1). To complete the induction, suppose the claim holds for \( m \leq n \) and consider the case
$m = n+1$. By way of contradiction, suppose the claim is false. Since $x$ must be selected from $S_X(B)$ or $S_X(B)$, suppose $UNE[S_X(B); P] = xM^p_c = UNE[S_X(C); P]$.

First, consider $S_X(B)$. Since $S_X$ is simple, so is $S_X(B)$. By persistence, every path to a terminal node $t$ below $B$ passes through a node $B^*_t$ (potentially $B$) with successors $B_t$ and $B \cap C$. Let $T(B, x) \equiv \{ t : x \in B_t \}$. There are two cases: (i) $T(B, x) = \emptyset$; and, (ii) $T(B, x) \neq \emptyset$. For both, $UNE[S_X(B \cap C); P] = x$. (i) Since $UNE[S_X(B); P] = x$ and $x \notin B$, for all $t$ below $B$, “backward induction” requires $UNE[S_X(B \cap C); P] = x$. (ii) Since $UNE[S_X(B); P] = x$, “backward induction” requires $UNE[S_X(B^*_t); P] = x$ for some $t \in T(B, x)$. Since $S_X$ is non-repetitive by Claim 1, $B^*_t \subset X$ so that $|B^*_t| \leq n$. Since $S_X(B^*_t)$ is simple, the induction hypothesis implies $UNE[S_X(B \cap C); P] = x$.

Next, consider $S_X(C)$. If $c \in B \cap C$, the same reasoning as in the last paragraph establishes $UNE[S_X(B \cap C); P] = c$, which is a contradiction. So, $c \in C \setminus B$. By persistence, every path to a terminal node $t$ below $C$ passes through a node $C^*_t$ (potentially $C$) with successors $C_t$ and $B \cap C$. Let $T(C, c) \equiv \{ t : c \in C_t \}$. By construction, $T(C, c) \neq \emptyset$. Since $UNE[S_X(C); P] = c$, “backward induction” requires $UNE[S_X(C_t); P] = c$ for some $t \in T(C, c)$. Since $UNE[S_X(B \cap C); P] = x$ and $xM^p_c$ however, $UNE[S_X(C^*_t); P] = x$. It then follows from “backward induction” that $UNE[S_X(C_t); P] \neq c$, which is a contradiction.

**Claim 12** If $S_X$ is a simple agenda where $B$ and $C$ are the successors of the root $X$, $UNE[S_X|B; P] = UNE[S_X(B); P]$.

**Proof.** If $B \cap C = \emptyset$, then $S_X|B = S_X(B)$ by definition and the claim follows. So, suppose $B \cap C \neq \emptyset$. Fix a profile $P \in P$ such that $UNE[S_X|B; P] = b$. By Remark 1, $UNE[S_X; P^B] = b$. Since $P$ and $P^B$ coincide on $B$, the proof is complete if $UNE[S_X(B); P^B] = b$. (In that case, $UNE[S_X(B); P] = b = UNE[S_X|B; P]$.) To establish $UNE[S_X(B); P^B] = b$, there are two cases: (i) $b \in B \setminus C$; and, (ii) $b \in B \cap C$. (i) Since $b \in B \setminus C$, “backward induction” requires $UNE[S_X(B); P^B] = b$. (ii) Since $UNE[S_X; P^B] = b \in B \cap C$, Claim 11 implies $UNE[S_X(B); P^B] = b$.

**Claim 13** If a decision rule $v$ is implementable by simple agenda, then it satisfies IS.

**Proof.** Suppose $v$ is implemented by $S_X$. The proof is by strong induction on $|X| = m$. Since $S_X$ is non-repetitive (by Claim 1), the base cases $m = 2, 3$ are trivial. To complete the induction, suppose the claim holds for $m \leq n$ and consider the case $m = n + 1$. Let $B$ and $C$ denote the two successors of the root $X$. To establish the induction step, fix an issue $A \notin X_1 \cup X_2$. Without loss of generality, there are two possibilities: (1) $A \not\subseteq B, C$; or, (2) $A \subseteq B$.

(1) In this case, $(B \cap A, C \cap A)$ splits $A$. To see this, it suffices to check conditions (i)-(iii) of the definition. (i) Since $B \cup C = X$, it follows that $(B \cap A) \cup (C \cap A) = (B \cup C) \cap A = X \cap A = A$. (ii) By way of contradiction, suppose that $B \cap A = (B \cap A) \cap (C \cap A)$. Then, $A = (B \cap A) \cap (C \cap A)$, so $A \subseteq C$ which is a contradiction. (iii) Note that $v(P, B) \equiv UNE[S_X|B; P] = UNE[S_X(B); P]$ and $v(P, C) \equiv UNE[S_X|C; P] = UNE[S_X(C); P]$ by Claim 12. Since $v(P, X) \equiv UNE[S_X; P]$ as well, “backward induction” requires $v(P, X) = v(P, \{v(P, B), v(P, C)\})$. Combined with Remark 1, this identity implies the following:

$$v(P, A) = v(P^A, X) = v(P^A, \{v(P^A, B), v(P^A, C)\}) = v(P^A, \{v(P^{B \cap A}, X), v(P^{C \cap A}, X)\})$$

$$= v(P^A, \{v(P, B \cap A), v(P, C \cap A)\}) = v(P, \{v(P, B \cap A), v(P, C \cap A)\}).$$

(2) By Claim 12, $v(P, B) \equiv UNE[S_X|B; P] = UNE[S_X(B); P]$. By Remark 1, it follows that $S_X(B)$ implements the restricted decision rule $v_B$ on $B$. Since $S_X$ is non-repetitive by Claim 1, $B \subset X$ so that $|B| \leq n$. Since $S_X$ is simple, so is $S_X(B)$. By the induction hypothesis, it then follows that $A$ is splittable for $v_B$. To conclude, suppose $(B_A, C_A)$ splits $A$ for $v_B$. Since $v_B(P, D) \equiv v(P, D)$ for all $D \in B$, it follows that $(B_A, C_A)$ also splits $A$ for $v$. ■

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**Proof of Theorem 1 (Necessity).** Let \( v \) denote the decision rule implemented by a simple agenda \( S_X \). By Claim 13, \( v \) satisfies IS. By the argument given in the text (after the statement of the theorem), \( v \) also satisfies ILA. \( \blacksquare \)

9.6.3 Proof of Uniqueness

**Note:** In this section, I abuse notation by referring to a node \( q \) of a simple agenda \( S_X \) by its label \( \ell(q) \).

As outlined in the text, the outcomes of \( v \) on Condorcet triples uniquely identify \( S_X^* \). The proof that \( S_X^* \) is the only simple agenda implementing \( v \) by strong induction on \(|X| = m\). The base cases \( m = 2, 3 \) follow from IS. To complete the induction, suppose the claim holds for \( m \leq n \) and consider the case \( m = n + 1 \). Fix a simple agenda \( S_X \) that implements \( v \); and, let \( B \) and \( C \) denote the two successors of the root \( X \) in \( S_X \).

First, note that \( (B, C) \) is the unique splitting of \( X \). Given Claim 8, it suffices to check conditions (i)-(iii) of the definition. (i) Since \( S_X \) is an agenda, \( B \cup C = X \). (ii) Since \( S_X \) is non-repetitive (by Claim 1), \( B, C \neq B \cap C \) (by Remark 12). (iii) By Claim 12, \( UNE[S_X|B; P] = UNE[S_X(B); P] \) and \( UNE[S_X|C; P] = UNE[S_X(C); P] \) for all \( P \in \mathbb{P} \). Since \( S_X \) implements \( v \), \( v(P, B) = UNE[S_X|B; P] \) and \( v(P, C) = UNE[S_X|C; P] \). So, \( v(P, B) = UNE[S_X(B); P] \) and \( v(P, C) = UNE[S_X(C); P] \). Since \( v(P, X) = UNE[S_X; P] \) as well, \( v(P, X) = v(P, \{v(P, B), v(P, C)\}) \).

To conclude, consider the restricted decision rules \( v_B \) and \( v_C \) on \( B \) and \( C \). As shown in the last paragraph, \( v_B(P, B) = UNE[S_X(B); P] \) and \( v_C(P, C) = UNE[S_X(C); P] \); for all \( P \in \mathbb{P} \). So, \( S_X(B) \) implements \( v_B \) (by Remark 1) and, similarly, \( S_X(C) \) implements \( v_C \). Since \( S_X \) is simple, so are \( S_X(B) \) and \( S_X(C) \). By the induction hypothesis, \( S_B^v \) (resp. \( S_C^v \)) is the unique simple agenda that implements \( v_B \) (resp. \( v_C \)). Consequently, \( S_B^v = S_X(B) \) and \( S_C^v = S_X(C) \). Since \( (B, C) \) splits \( X \) (as shown in the last paragraph), it then follows from the definition of \( S_X^* \) that \( S_X = S_X^* \).

9.7 Proof of Propositions 1 and 2

**Proof of Proposition 1.** Consider a decision rule \( v \) that is implementable by simple agendas. It suffices to show that \( v \) is preference monotonic at \( X \). Since \( v(P, A) = v(P^X, X) \) by Remark 1, the result follows for \( A \subseteq X \). The proof is by strong induction on \(|X| = m\). The base cases \( m = 2, 3 \) are trivial.

To complete the induction, suppose the claim holds for \( m \leq n \) and consider the case \( m = n + 1 \). Let \( S_X \) denote a simple agenda that implements \( v \); and, let \( B \) and \( C \) denote the successors of the root \( X \). For a profile \( P \) such that \( v(P, X) \equiv UNE[S_X; P] = x \), there are two cases: (i) \( x \in B \cup C \); and, (ii) \( x \in B \setminus C \).

(i) By Claim 11, \( UNE[S_X|B; P] = UNE[S_X(C); P] = x \). Since \( S_X \) is non-repetitive by Claim 1, \( B, C \subseteq X \) so that \(|B|, |C| \leq n \). Since \( S_X \) is simple, so are \( S_X(B) \) and \( S_X(C) \). So, \( UNE[S_X(B); P^{\uparrow x}] = x \) and \( UNE[S_X(C); P^{\uparrow x}] = x \) by the induction hypothesis. Then, “backward induction” requires \( v(P^{\uparrow x}, X) \equiv UNE[S_X; P^{\uparrow x}] = x \). (ii) By the same reasoning as case (i), \( UNE[S_X(B); P^{\uparrow x}] = UNE[S_X(B); P] = x \). Moreover, \( UNE[S_X(C); P^{\uparrow x}] = UNE[S_X(C); P] \) since \( x \notin C \). Then, “backward induction” requires \( v(P^{\uparrow x}, X) \equiv UNE[S_X; P^{\uparrow x}] = x \). \( \blacksquare \)

**Proof of Proposition 2.** This follows directly from the discussion in the text. \( \blacksquare \)

9.8 Proof of Theorem 2 (and Proposition 3)

Subsections 1 and 2 establish sufficiency and necessity for Theorem 2, respectively.

9.8.1 Proof of Sufficiency

**Claim 14** Suppose \( v \) satisfies IS and ILA. If \( a^* \) is marginal for \( a \notin X_1 \cup X_2 \) and \( x \in A \setminus a^* \), then \( a^* \) is marginal for \( A \setminus x \).
Proof. Fix some \( x \in A \setminus a^* \). By way of contradiction, suppose \( v(P, A \setminus x) = a^* \) for some profile \( P \) where \( a^* \) is not the Condorcet winner in \( A \setminus x \). By Claim 4, \( v(P_x, A \setminus x) = v(P, A \setminus x) \) for any profile \( P_x \) that coincides with \( P \) except \( x \) is demoted to Condorcet loser in \( A \). Moreover, \( v(P_x, A) = v(P_x, A \setminus x) \) by ILA. So, \( v(P_x, A) = a^* \), which contradicts the assumption that \( a^* \) is marginal for \( A \). ■

Claim 15 If \( v \) satisfies IS and ILA, then every \( A \not\in X_1 \cup X_2 \) has at most two marginal alternatives.

Proof. Suppose otherwise. Denote any three marginal alternatives in \( A \) by \( x, y, \) and \( z \) and consider the Condorcet triple \( P_{xyz} \) (as defined in the text). Then, \( v(P_{xyz}, \{x, y, z\}) \notin \{x, y, z\} \) by Claim 14, which is a contradiction. ■

Claim 16 Suppose \( v \) satisfies IS and ILA. If \( (B, C) \) splits \( A \not\in X_1 \cup X_2 \) and \( a^* \) is marginal for \( A \), then:

(i) If \( a^* \in C \setminus B \), then \( (B, C) = (b, A \setminus b) \); and, (ii) if \( a^* \in B \cap C \) and \( a^{**} \) is also marginal for \( A \), then \( a^{**} \in B \cap C \).

Proof. (i) By way of contradiction, suppose \( |B \setminus C| \geq 2 \). Fix \( b, b' \in B \setminus C \) and consider the triple \( P_{a \ast bb'} \). By Claim 6, \( v(P_{a \ast bb'}, \{a^*, b, b'\}) = v(P_{a \ast bb'}, \{a^*, b, b'\}, a^*) \) = \( a^* \). By Claim 14, this contradicts the assumption that \( a^* \) is marginal for \( A \). (ii) By way of contradiction, suppose \( a^{**} \in C \setminus B \). Fix some \( b \in B \setminus C \) and consider the triple \( P_{a \ast ba^{**}} \). Then, a contradiction obtains along the same lines as (i). ■

(a) Proof of Lemma 1 (Revealed Priority in General)

Claim 17 If \( v \) satisfies IS, ILA, and \( M \), then \( \succcurlyeq_v \) is asymmetric and \( \succeq_v \) is complete.

Proof. By definition of \( \sim_v \), the completeness of \( \succeq_v \) follows from the asymmetry of \( \succcurlyeq_v \). To see that \( \succcurlyeq_v \) is asymmetric, suppose \( y \succcurlyeq_v z \) and \( z \succcurlyeq_v y \) for some \( y, z \in X \); and, let \( Y \) and \( Z \) denote issues that reveal \( y \succcurlyeq_v z \) and \( z \succcurlyeq_v y \), respectively. The proof that this amounts to a contradiction is by induction on \( |Y \cup Z| \).

For the base case \( |Y \cup Z| = 4 \), it must be that \( Y = \{a^*, y, z\} \) and \( Z = \{x, y, z\} \) for alternatives \( a^*, x \in X \). (The possibility that \( Y = \{y, z\} \) is ruled out by ILA; and, the possibility that \( Z \subseteq Y \) is ruled out by Claim 14.) By Claim 14, neither \( y \) nor \( z \) can be marginal for \( Y \cup Z \). Since some alternative must be marginal for \( Y \cup Z \) by M, suppose, without loss of generality, that \( a^* \) is marginal. Then, \( a^* \) is also marginal for \( Y \) by Claim 14. Now, consider the splitting \( (B, C) \) of \( \{a^*, x, y, z\} \). There are two cases: (i) \( a^* \in C \setminus B \); and, (ii) \( a^* \in B \cap C \).

(i) By Claim 16(i), \( (B, C) = (b, \{a^*, x, y, z\} \setminus b) \) with \( b \neq a^* \). By Claim 6, this is a contradiction: if \( b = x \), then \( z \) is marginal for \( Z \); if \( b = y \), then \( y \) is not marginal for \( Z \); and, if \( b = z \), then \( z \) is not marginal for \( Y \).

(ii) Since \( a^* \) and \( z \) are marginal for \( Y \), \( v(P_{a \ast yz}, Y) = y \). There are two cases: (a) \( z \in C \setminus B \); and, (b) \( z \in B \cap C \).

(a) There are two sub-cases: \( y \in B \setminus C \); and, \( y \in C \). In the first case, \( (\{a^*, z\}, \{a^*, y\}) \) splits \( Y \) by Claim 6. So, \( z \) is not marginal for \( Y \), a contradiction. In the second case, \( v(P_{a \ast yz}, Y) = v(P_{a \ast yz}, \{v(P_{a \ast yz}, B \cap Y), v(P_{a \ast yz}, Y)\}) = v(P_{a \ast yz}, \{a^*, y\}) \neq y \) by Claim 6, a contradiction. (b) In this case, \( (B, C) = (\{a^*, y, z\}, \{a^*, x, z\}) \). By Claim 6, \( (\{y, z\}, \{x, z\}) \) splits \( Z \). So, \( y \) is not marginal for \( Z \), which is a contradiction.

To complete the inductive hypothesis, suppose the claim holds for \( |Y \cup Z| = n \) and consider the case \( |Y \cup Z| = n+1 \). By M and Claim 15, \( Y \) and \( Z \) each have one or two marginal alternatives. If both have two, this reduces to the case \( |Y \cup Z| = n \) by Claim 14. (Without loss of generality, \( |Y| \geq 4 \). So, \( Y \) must contain some non-marginal alternative that is not marginal in \( Z \).) If both sets have one marginal alternative, then either \( y \) or \( z \) is marginal for \( Y \cup Z \) by M and Claims 14-15. So, either \( y \) is marginal for \( Y \) or \( z \) is marginal for \( Z \) by Claim 14, both contradictions. So, suppose \( a^* \) and \( z \) are marginal for \( Y \); and, \( y \) is marginal for \( Z \). By M and Claim 14, \( a^* \notin Z \) is the only marginal alternative in \( Y \cup Z \). By Claim 14: \( a^* \) and \( z \) are marginal for \( \{a^*, y, z\} \); and, \( y \) is marginal for \( \{x, y, z\} \) for any \( x \in Z \).
Now, consider the splitting \((B, C)\) of \(Z^* \equiv Z \cup a^*\). There are two cases: (i) \((B, C) = (b, Z^* \setminus b)\) with \(b \neq a^*\); and, (ii) \(a^* \in B \cap C\). For both, \(y\) and \(z\) must appear in the same sub-issues as \(a^*\). (i) By the argument in (i) of the base case, \(b \neq y, z\). So, \(\{y, z\} \subseteq C\) as claimed. (ii) By the argument in (ii)(a) of the base case, \(z \in B \cap C\). In turn, this implies \(y \in B \cap C\). To see why, suppose \(y \in B \setminus C\) and fix some \(x \in C \setminus B\). Then, \(\{(y, z), (x, z)\}\) splits \(\{x, y, z\}\) by Claim 6. By the argument in (ii)(b) of the base case, this is a contradiction. So, \(\{y, z\} \subseteq B \cap C\) as claimed.

Continuing in this vein on the sub-issues \(B\) and \(C\), \(y\) and \(z\) always appear in the same sub-issues (up to the splitting of \(\{a^*, y, z\}\)). Accordingly, \(y\) and \(z\) appear in exactly the same subgames of \(S^*_y\) (i.e., \(S^*_z\) after \(a^*\) is pruned). By assumption, \(v(P, Z) = y\) for some profile \(P\) where \(y\) is not the Condorcet winner in \(Z\). Since \(S^*_y\), implements \(v\) on \(Z^*\) by Theorem 1, \(v(P, Z) = UNE[S^*_{y,z}^*; P] = y\). To see the contradiction, consider the profile \(P^a\) that permutes \(z\) and \(y\) in every voter’s preference. From the symmetry of \(S^*_y\), \(v(P^a, Z) = UNE[S^*_{y,z}^*; P^a] = z\). But, this contradicts that \(z\) is marginal for \(Z\) and establishes that \(\succ_{\nu}\) is asymmetric. ■

Claim 18 If \(v\) satisfies IS, ILA, and \(M\), then \(\succ_{\nu}\) defines a priority.

Proof. Given Claim 17, it suffices to show that: (i) the indifference classes of \(\sim_{\nu}\) contain one or two alternatives; and, (ii) \(\succ_{\nu}\) is transitive. (i) By way of contradiction, suppose \(x \sim_{\nu} y \sim_{\nu} z\). By M, \(\{x, y, z\}\) has a marginal alternative. By definition of \(\sim_{\nu}\), it follows that \(\{x,y,z\}\) has three marginal alternatives, which contradicts Claim 15. (ii) Given (i), it suffices to rule out the possibility that \(x \succ_{\nu} y \succ_{\nu} z \succ_{\nu} x\). Since \(\succ_{\nu}\) is asymmetric (by Claim 17) and \(\{x,y,z\}\) has a marginal alternative (by M), this leads to a contradiction along the lines of (i). ■

Proof of Lemma 1. Claim 18 establishes (i). To establish (ii), fix an issue \(A\). By M, some \(x \in A\) is marginal. Let \(z \in \min_{\sim_{\nu}} A\) and \(y \equiv \min_{\succ_{\nu}} A \setminus z\). If \(x \in A \setminus \{y, z\}\), one obtains a contradiction along the lines of Claim 18. ■

(b) Proof of Corollary 1 (Revealed Priority via Condorcet Triples)

Claim 19 Suppose \(v\) satisfies IS, ILA, and \(M\). If \(v(B, C)\) splits \(A \not\in X_1 \cup X_2\) and \(|C \setminus B| \geq 2\), then \(|B \setminus C| = 1\).

Proof. By way of contradiction, suppose \(|C \setminus B|, |B \setminus C| \geq 2\). Fix any \(b, b' \in B \setminus C\) and \(c, c' \in C \setminus B\). By Claim 6, \(\{(b, b'), (c, c')\}\) splits \(A' \equiv \{b, b', c, c'\}\). Given this splitting, there exists some profile \(P_a\) where \(a \in A'\) is not the Condorcet winner such that \(v(P_a, A') = a\). So, \(A'\) has no marginal alternatives, which contradicts M. ■

For a weak ordering \(\succeq\) on \(X\), let \(\succ^a\) denote any linear ordering such that \(y \succ^a \succ z\) implies \(y \succ^a z\) for all \(y, z \in X\). Given an issue \(A = \{a_1, \ldots, a_K\} \in X\) labeled according to \(\succ^a\) (i.e., \(a_i \succ^a a_k\) if \(i < k\)), let

\[
A^j_i = \begin{cases} 
\{a_j, \ldots, a_k\} & \text{if } j \leq k \leq K \\
\emptyset & \text{otherwise.} 
\end{cases}
\]

Claim 20 Suppose \(v\) satisfies IS, ILA, and \(M\). If \(A = \{a_1, \ldots, a_K\}\) is labeled according to \(\succ^a\) for \(K \geq 2\), then:

\(a_1 \cup A^K_j, A^K_j\) splits \(A\) for some \(j\) such that \(j \in \{3, \ldots, K + 1\}\).

Proof. By Claim 16(i) and Lemma 1, there are two possibilities for the splitting \((B, C)\) of \(A\): (i) \((B, C) = (b, A \setminus b)\) with \(b \neq a_k\); or, (ii) \(a_k \in B \cap C\). In either case, I show that \((B, C)\) has the desired structure.

(i) I show \(b = a_1\). By way of contradiction, suppose \(b = a_k\) for some \(k \neq 1, K\). Then, \(v(P_{a_k, a_1, a_k, a_1}) = a_k\) by Claim 6 so that \(a_k \succ_{\nu} a_1\) by Lemma 1. Since \(a_1 \succ_{\nu} a_k\) by assumption, \(a_k \succ_{\nu} a_1\) contradicts the fact that \(\succ_{\nu}\) is a weak ordering (by Claim 17). So, \(b = a_1\) as required.
Proof. (i) If \( v \) satisfies IS, ILA, and M, then:

(i) \( y \succ_v z \) if and only if there exists an \( x \in X \) such that \( v(P_{xyz}, \{x, y, z\}) = y \); and,

(ii) \( y \sim_v z \) if and only if, for every \( x \in X \setminus \{y, z\} \),

\[
\begin{align*}
&v(P_{xyz}, \{x, y, z\}) = z \text{ and } v(P_{xzy}, \{x, y, z\}) = y \\
or\ &v(P_{xyz}, \{x, y, z\}) = x \text{ and } v(P_{xzy}, \{x, y, z\}) = x
\end{align*}
\]

By Claim 6, \( v \) satisfies IS for \( x, y, z \). This establishes that \( z \) is marginal for \( \{x, y, z\} \) and \( y \) is not. So, \( y \succ_v z \).

(ii) For \( |X| = 3 \), the claim is trivial. The claim is also trivial if \( |X| \geq 4 \) and:

(a) \( y \succ_v x \supseteq z \) for some \( x \in X \setminus \{y, z\} \);

or, (b) \( z \) and \( w \) are marginal for some \( Y \) such that \( y \in Y \).

(a) By Lemma 1, \( y \) cannot be marginal for \( \{x, y, z\} \). Then, \( v(P_{xyz}, \{x, y, z\}) = y \) from Table 1.

(b) By Claim 14, \( z \) and \( w \) are marginal for \( \{y, z, w\} \). So, \( y \) cannot be marginal for \( \{x, y, z\} \) by Claim 15. Then, \( v(P_{xyz}, \{x, y, z\}) = y \) from Table 1.

So, suppose \( |X| \geq 4 \) and conditions (a)-(b) do not hold. By way of contradiction, suppose \( v(P_{xyz}, \{x, y, z\}) \in \{x, z\} \) for all \( x \in X \). Now, fix an \( A = \{x, x', y, z\} \) where \( x, x' \succ_v y \); and, consider the splitting \((B, C)\). By Claim 20 \((B, C) = (x \cup B', \{x', y, z\}) \) for \( B' \subseteq \{y, z\} \) (where \( y \in B' \) implies \( z \in B' \)). If \( z \in B' \) and \( y \notin B' \), then \( \{x, z, \{y, z\}\} \) splits \( \{x, y, z\} \) by Claim 6. So, \( v(P_{xyz}, \{x, y, z\}) = y \), a contradiction.

In the first case:

\[
\begin{align*}
v(P_{xx'y}, \{x, x', y\}) &= v(P_{xx'z}, \{x, x', z\}) \quad \text{and} \quad v(P_{x'xy}, \{x, x', y\}) = x = v(P_{x'xz}, \{x, x', z\}).
\end{align*}
\]

In the second case:

\[
\begin{align*}
v(P_{xx'y}, \{x, x', y\}) &= v(P_{xx'z}, \{x, x', z\}).\end{align*}
\]

To complete the proof, consider the agenda \( S'_U \) on \( U = \{x : x \succ_v y\} \cup \{y, z\} \). From the observations in the previous paragraph, \( y \) and \( z \) appear in the same subgames of \( S'_U \). By assumption, there is some \( Y \subseteq X \) that leads to the inference \( y \succ_v z \). Since conditions (a)-(b) do not hold, Lemma 1 implies that \( Y \subseteq U \). Then, a contradiction obtains by the symmetry argument given at the end of Claim 17. In particular, \( v(P, Y) = UNE[S'_{U \Delta z}; P'] \neq z \) iff \( v(P', Y) = UNE[S'_{U \Delta z}; P'] \neq y \). Since \( z \) is marginal on \( Y \), it follows that \( y \) is marginal on \( Y \).

(ii) The result follows from (i), the potential splittings of \( \{x, y, z\} \), and the fact that \( \succ_v \) is a weak ordering.

(c) Proof of Lemma 2 (Revealed Amendment Rule)

Definition 14 Given a decision rule \( v \) with revealed priority \( \succ_v \), \( x \in X \) is revealed to amend \( b \in X \) if:

\[
(1) b \succ_v x; \text{ and, (2) } v(P_{xbc}, \{b, c, x\}) = c \text{ for all } c \in X \text{ such that } b \succ_v c \succ_v x.
\]

Then, define \( \alpha_v \) as follows: \( b \in \alpha_v(x) \) if \( x \) is revealed to amend \( b \) and \( x \) is not revealed to amend any \( a \succ_v b \).

Lemma 2 If \( v \) satisfies IS, ILA, and M, then \( (\succ_v, \alpha_v) \) defines an admissible pair.
Proof. It suffices to check that \((\succeq_v, \alpha_v)\) is admissible.

(i) if \(z \in X_1\), then \(\alpha(z) = \emptyset\). By Definition 14.

(ii) if \(z \in X_j\) for \(j \neq 1\), then \(\alpha(z) \subseteq X_k\) for some \(k < j\): By Definition 14, it suffices to check that \(z\) amends some alternative. By Lemma 2, there are two possibilities: a unique \(b\) has immediately higher priority than \(z\); or, distinct \(b\) and \(c\) have immediately higher priority than \(z\). In the first case, \(b \succ v z\) implies \(z\) amends \(b\) (since (2) is vacuous). In the second, \(b \sim v c \succ v z\) implies \(v(P_{zbc}, \{b, c, z\}) = c\) by Corollary 1. So, again, \(z\) amends \(b\).

(iii) if \(z, z' \in X_j\) for \(j \neq 1\), then \(\alpha(z) = \alpha(z')\): By Definition 14, it suffices to show that \(z'\) is revealed to amend \(b\) if \(z\) is revealed to amend \(b\). By way of contradiction, suppose: \(z\) is revealed to amend \(b\); and, \(z'\) is not. By Definition 14, there exists some \(c\) such that \(b \succeq v c \succ v z' \sim v z\) and \(v(P_{zbc}, \{b, c, z'\}) \neq c\). By Corollary 1, \(v(P_{zbc}, \{b, c, z\}) \neq z'\). Otherwise, \(z' \succ v b\) which contradicts that \(\succeq_v\) is a weak ordering (by Claim 17). So, \(v(P_{zbc}, \{b, c, y\}) = b\) by Corollary 1, \(b \succ v c\). Since \(z\) is revealed to amend \(b\), Definition 14 implies \(v(P_{zbc}, \{b, c, z\}) = c\). To summarize, \(v(P_{zbc}, \{b, c, z\}) = c\) and \(v(P_{zbc}, \{b, c, z\}) = b\) for some \(c\) such that \(b \succ v c \succ v z' \sim v z\).

By Claim 20, the splitting of \(\{b, c, z, z'\}\) is \((b \cup B', \{c, z, z'\})\) for \(B' \subseteq \{z, z'\}\). By Claim 6, \(v(P_{zbc}, \{b, c, z\}) = c\) implies \(z \in B'\) and \(v(P_{zbc}, \{b, c, z\}) = b\) implies \(y \notin B'\). So, the splitting of \(\{b, c, z, z'\}\) is \((\{b, z\}, \{c, z, z'\})\). By Claim 6, this implies \(v(P_{zbx}, \{b, z, z'\}) = y\) so that \(z' \succ v z\) by Corollary 1, which is a contradiction.

(iv) if \(z \in X_{j+1}\) and \(|X_j| = 2\), then \(\alpha(z) = X_j\) or \(\alpha(z) \subseteq X_k\) for some \(k < j\): By Definition 14 and (i) above.

\((\) Proof\(\) of Sufficiency\(\))

Note: In this section, I refer to a node \(q\) of a priority (resp. simple) agenda \(Pr_X\) (resp. \(S_X\)) by its label \(\ell(q)\).

For a weak ordering \(\succeq\) on \(X\), let \(L_{\succeq}(a) = \{x \in X : a \succ x\}\) denote the strict lower contour set of \(a \in X\).

**Claim 21** Fix a priority agenda \(Pr_{(\succeq, \alpha)}\) and a non-terminal node \(A = \{a_1, ..., a_K\}\) labeled according to \(\rightarrow^*\).

Then, the successors of \(A\) are \(a_1 \cup A^K_j\) and \(A^K_j\) for some \(j \in \{3, ..., K + 1\}\). What is more:

(i) \(A^K_j = X^m_{m-K+2}\) where \(X = \{x_1, ..., x_m\}\) is labeled according to \(\rightarrow^*\); and,

(ii) \(a_2 \succ a_j\) and \(a_j\) is a highest priority alternative in \(A^K_j\) that amends \(a_1\) (if such an alternative exists).

**Proof.** Let \(B\) and \(C\) denote the the two successors of node \(A\) in \(Pr_{(\succeq, \alpha)}\). Since \(\succeq\) is a priority, there are two cases: (1) \(\max_{\succeq} A = \{a_1, a_2\}\); or, (2) \(\max_{\succeq} A = \{a_1\}\). Since the claim is trivial if \(|A| = 2\), suppose \(|A| \geq 3\).

(1) By construction (step (ii) in Class 3), \(a_1\) and \(a_2\) must have been added at \(A\). Since they must be added to different successors of \(A\), \(a_1 \in B \setminus C\) and \(a_2 \in C \setminus B\). By definition of \(\alpha\), the next highest priority alternative(s) in \(X\) are added under the successors identified with \(a_1\) and \(a_2\). By a straightforward induction argument, it follows that \(B = a_1 \cup L_{\succeq}(a_1)\) and \(C = a_2 \cup L_{\succeq}(a_2)\). So, \(B = a_1 \cup A^K_j\), \(C = A^K_j = X^m_{m-K+2}\), and \(a_2 \succ a_3\).

(2) By construction (step (i) in Class 3), \(a_1\) and \(a_2\) and the next highest priority alternative(s) at \(A\) must be added to different successors of \(A\). Letting \(A_{-1} = \max_{\succeq}(A \setminus a_1)\), it follows that \(a_1 \in B \setminus C\) and \(A_{-1} \subseteq C \setminus B\). So, \(C = A_{-1} \cup L_{\succeq}(a_2) = A^K_j = X^m_{m-K+2}\) by the same argument as (1). Similarly, \(B = a_1 \cup A^K_j\) where \(a_j\) is a highest priority in \(C = A^K_j = X^m_{m-K+2}\) that amends \(a_1\) besides \(a_2 \succ a_j\) (if such an alternative exists). ■

**Proof of Theorem 2 (Sufficiency).** Fix a decision rule \(v\) that satisfies IS, ILA, and M.

First, note that \(\succeq_v\) defines a priority (by Lemma 1) and \(\alpha_v\) defines an amendment rule consistent with \(\succeq_v\) (by Lemma 2). So, \((\succeq_v, \alpha_v)\) defines a priority agenda \(Pr_X^v\). By the sufficiency portion of Theorem 1, \(Pr_X^v\) implements \(v\) if
\( \mathcal{P}_r^X = S_X^X \). To establish this, I show that the successors at the root nodes of \( \mathcal{P}_r^X \) and \( S_X^X \) coincide. Extending this line of reasoning by a simple induction argument, it follows that \( \mathcal{P}_r^X = S_X^X \).

To see that the successors at the (common) root \( X \) of \( \mathcal{P}_r^X \) and \( S_X^X \) coincide, label \( X = \{x_1, ..., x_m\} \) according to \( \succ^* \). By Claim 20, the successors of \( X \) in \( S_X^X \) are \( x_1 \cup X_1^m \) and \( X_2^m \) for some \( j \) such that \( j \in \{3, ..., m+1\} \). There are two cases to consider: (i) \( j = m + 1 \) (so that \( X_j^m = \emptyset \)); and, (ii) \( j \in \{3, ..., m\} \) (so that \( X_j^m \neq \emptyset \)).

In both cases, the fact that the successors of \( X \) on \( \mathcal{P}_r^X \) coincide with the successors on \( S_X^X \) follows from Claim 21: (i) Claim 6 applied to \( (x_1, X_1^m) \) gives \( v(P_{x_1,j}, \{x_1, x_j, c\}) = x_1 \) for all \( x_j, c \) such that \( x_1 \succeq c \succ x_j \). So, no \( x_j \) such that \( x_2 \succ x_j \) is revealed to amend \( x_1 \) (by Definition 14). (ii) Claim 6 applied to \( (x_1 \cup X_1^m, X_2^m) \) gives \( v(P_{x_1,j}, \{x_1, x_j, x_k\}) = x_1 \) and \( v(P_{x_1,j}, \{x_1, x_k, x_j\}) = x_k \) for all \( x_k \in X_{j-1}^m \). So, \( x_k \succ x_j \) (by Corollary 1) and \( x_j \) is revealed to amend \( x_1 \). To see that no \( x_k \) such that \( x_2 \succ x_k \succ x_j \) is revealed to amend \( x_1 \), observe that \( v(P_{x_1,j}, \{x_1, x_2, x_k\}) = x_1 \) for all \( x_k \in X_{j-1}^m \).

9.8.2 Proof of Necessity (and Proposition 3)

Note: In this section, I abuse notation by referring to a node \( q \) of a priority agenda by its label \( \ell(q) \).

**Claim 22** Fix a priority agenda \( \mathcal{P}_{(\succeq, \alpha)} \) on \( X = \{x_1, ..., x_m\} \) (labeled according to \( \succ^* \)) such that \( x_j \) amends \( x_1 \) and \( x_2 \) (according to \( \alpha \)). Then, every path from \( X \) to a terminal node passes through a non-terminal node with a successor \( X_j^m \).

**Proof.** The proof is by strong induction on \(|X| \equiv m\). The base case \( m = 3 \) is trivial.

To complete the induction, suppose the claim holds for \( m \leq n \) and consider the case \( m = n + 1 \). By Claim 21, the two successors of the root \( X \) are \( x_1 \cup X_1^m \) and \( X_2^m \) for \( k \in \{3, ..., m+1\} \). Moreover, \( k \neq m + 1 \). Otherwise, no alternative amends \( x_1 \) and \( x_2 \). Since path from \( X \) to a terminal node passes through \( x_1 \cup X_1^m \) or \( X_2^m \), it suffices to show that every path from \( x_1 \cup X_1^m \) or \( X_2^m \) to a terminal node passes through a non-terminal node with the desired characteristics. There are three cases to consider: (i) \( j = k = 3 \); (ii) \( j = k > 3 \); and, (iii) \( j \in \{k + 1, ..., m\} \).

(i) By Claim 21, \( X_3^m \) is a successor of \( x_1 \cup X_1^m \) and \( X_2^m \), making these the nodes with the desired characteristics.

(ii) By the same reasoning as case (i), every path from \( x_1 \cup X_1^m \) to a terminal node passes through a non-terminal node with the desired characteristics. Next, consider \( X_2^m \). By definition of \( \alpha \) and the assumption that \( x_j \) amends \( x_1 \) and \( x_2 \), it follows that \( x_j \) also amends \( x_3 \). By construction, \( \mathcal{P}_{X^m} \) is a priority agenda on \( n \) or fewer alternatives where \( x_j \) amends the two top-priority alternatives. By the induction hypothesis, every path from \( X_2^m \) to a terminal node passes through a non-terminal node with the desired characteristics.

(iii) As in case (ii), \( x_j \) amends \( x_3 \). So, \( x_j \) also amends \( x_k \) by definition of \( \alpha \). As in case (ii), every path from \( X_2^m \) to a terminal node passes through a non-terminal node with the desired characteristics. By similar reasoning, every path from \( x_1 \cup X_1^m \) to a terminal node also passes through a non-terminal node with the desired characteristics.

**Claim 23** Every priority agenda is persistent.

**Proof.** The proof is by strong induction on \(|X| \equiv m\). The base cases \( m = 2, 3 \) are straightforward. To complete the induction, suppose the claim holds for \( m \leq n \) and consider the case \( m = n + 1 \).

Fix a priority agenda \( \mathcal{P}_X \) defined by \( (\succeq, \alpha) \); and, label \( X = \{x_1, ..., x_m\} \) according to \( \succ^* \). By Claim 21, the two successors of the root \( X \) are \( x_1 \cup X_1^m \) and \( X_2^m \) for \( k \in \{3, ..., m+1\} \). By construction, \( \mathcal{P}_X(x_1 \cup X_1^m) \) and \( \mathcal{P}_X(X_2^m) \) are priority agendas on \( n \) or fewer alternatives. From the induction hypothesis, it follows that these agendas are persistent. So, \( \mathcal{P}_X \) is persistent at every non-terminal node except possibly \( X \). To check persistence at \( X \), there are two cases to consider: (i) \( k \in \{3, ..., m\} \); and, (ii) \( k = m + 1 \).
The uncontested alternatives at $X$ are $X^m$. By definition, $x_k$ amends $x_1$ and $x_2$. By Claim 22, it then follows that every path from $X$ to a terminal node in $Pr_X$ passes through a non-terminal node with successor $X_k^m$. So, $Pr_X$ is persistent at $X$. (ii) There are no uncontested alternatives at $X$. So, $Pr_X$ is trivially persistent at $X$. $lacksquare$

Proof of Remark 5. (i) History-independence follows by construction; and, persistence by Claim 23. (ii) By way of contradiction, suppose that the right-hand agenda in Example 3 is a priority agenda. (In fact, the argument is identical for the left-hand agenda.) Since $x_3$ and $x_4$ must have the highest priority (requirement (i) of admissibility). Since $x_3$ and $x_4$ amend different alternatives, they cannot have equal priority (requirement (iii) of admissibility). From these two observations, $x_3$ and $x_4$ must then have consecutive priority. But this is a contradiction (requirement (ii) of admissibility): $x_3$ does not amend $x_4$; and, $x_4$ does not amend $x_3$. $lacksquare$

Proof of Proposition 3. Suppose $Pr_X$ implements $\nu$ where $Pr_X$ is defined by $(\succeq_\alpha)$ and $X = \{x_1, \ldots, x_m\}$ is labeled according to $\succ^*$. To establish the result, it suffices to show that $x_m$ is marginal for $X$. By Remark 1, it then follows that $x_m$ is marginal for any issue $A \subseteq X$ such that $x_m \in A$. Since the pruned agenda $Pr_X\{x_1, \ldots, x_m\}$ is a priority agenda on $X \setminus x_m$ such that $x \succeq x_m$ for all $x \in X \setminus x_m$, the same argument establishes that $x_m$ is marginal for any issue $A \subseteq X \setminus x_m$ such that $x_m$ is marginal for any issue $A \subseteq X \setminus x_m$. By progressively deleting the highest index alternative (among those remaining) and repeating this argument, it follows that there exists a marginal alternative for all $A \notin X_1 \cup X_2$.

The proof that $x_m$ is marginal for $X = \{x_1, \ldots, x_m\}$ is by strong induction on $|X| = m$. The base cases $m = 2, 3$ are trivial. To complete the induction, suppose the claim holds for $m \leq n$ and consider the case $m = n + 1$. By Claim 21, the successors of the root $X$ are $x_1 \cup X_k^m$ and $X_2^m$. There are two cases: (i) $k = m + 1$; and, (ii) $k \in \{3, \ldots, m\}$.

(i) By construction, the root splits $Pr_X$ into $x_1$ and a priority agenda $Pr_X(X_2^m)$ on $n$ alternatives. Since $x_i \succeq x_m$ for all $x_i \in X_2^m$, the induction hypothesis implies $UNE[Pr_X(X_2^m); P] = x_m$ only if $x_m$ is the Condorcet winner on $(P, X_2^m)$. Then, “backward induction” implies that $UNE[Pr_X; P] = x_m$ only if $x_m$ is the Condorcet winner on $(P, \{x_1, x_m\})$ as well. Combining the last two observations gives the result.

(ii) By construction, the agendas $Pr_X(x_1 \cup X_k^m)$ and $Pr_X(X_2^m)$ are priority agendas on $n$ or fewer alternatives. By Claim 11, $UNE[Pr_X; P] = x_m$ only if $UNE[Pr_X(x_1 \cup X_k^m); P] = x_m$ and $UNE[Pr_X(X_2^m); P] = x_m$. Since $x_i \succeq x_m$ for all $x_i \in X$, the induction hypothesis implies that this is the case only if $x_m$ is the Condorcet winner on $(P, x_1 \cup X_k^m)$ and $(P, X_2^m)$. Combining the last two observations gives the result. $lacksquare$

Proof of Theorem 2 (Necessity). Fix a decision rule $\nu$ implemented by a priority agenda $Pr_X$. Given Remark 5, the necessity portion of Theorem 1 establishes that $\nu$ satisfies IS and ILA. Moreover, $\nu$ satisfies $M$ by Proposition 3. $lacksquare$

9.9 Proof of Proposition 4

Note: In this section, I abuse notation by referring to a node $q$ of a priority agenda $Pr_X$ by its label $\ell(q)$.

Suppose $Pr_X$ is defined by $(\succeq_\alpha)$ where $X = \{x_1, \ldots, x_m\}$ is labeled according to $\succ^*$. The proof is by strong induction on $|X| = m$. The base cases $m = 2, 3$ are trivial. To complete the induction, suppose the claim holds for $m \leq n$ and consider the case $m = n + 1$.

Fix a profile $P$ such that $UNE[Pr_X; P] = x_k$. (Where $x_{k-1} \sim x_k$, suppose that every alternative that amends $x_{k-1}$ also amends $x_k$.) By Claim 21, the successors of the root $X$ are: $B \equiv x_k \cup X_j^m$ for some $j \in \{3, \ldots, m + 1\}$; and, $C \equiv X_2^m$. So, there are four cases to consider: (i) $k \in \{j + 1, \ldots, m\}$; (ii) $k \in \{3, \ldots, j - 1\}$; (iii) $k = j$; and, (iv) $k = 2$. To establish the result, I show that $UNE[Pr_X; P] = x_k$ in each case (where $Pr_X \equiv Pr_X^\alpha$ to simplify the notation):
(i) Since \(x_k \in B \cap C\) and \(UNE[Pr_X; P] = x_k\), Claim 11 implies \(UNE[Pr_X(B); P] = UNE[Pr_X(C); P] = x_k\). Since \(Pr_X(B)\) and \(Pr_X(C)\) are priority agendas on \(n\) or fewer alternatives and \(k \neq j\), the induction hypothesis implies \(UNE[Pr_X^k(B); P] = UNE[Pr_X^k(C); P] = x_k\). So, \(UNE[Pr_X^k; P] = x_k\) as required.

(ii) Since \(x_k \notin B\) and \(UNE[Pr_X; P] = x_k\), “backward induction” implies \(UNE[Pr_X(C); P] = x_k\). Since \(Pr_X(C)\) is a priority agenda on \(n\) or fewer alternatives, the induction hypothesis implies \(UNE[Pr_X^k(C); P] = UNE[Pr_X(C); P] = x_k\). Since \(x_k \notin B\), \(Pr_X^k(B) = Pr_X(B)\). So, “backward induction” implies \(UNE[Pr_X^k; P] = x_k\) as required.

(iii) By the same reasoning as in case (i), \(UNE[Pr_X(B); P] = UNE[Pr_X(C); P] = x_j\) and \(UNE[Pr_X^k(C); P] = x_j\). By way of contradiction, suppose \(UNE[Pr_X^k; P] \neq x_j\). Then, \(UNE[Pr_X^k; P] = x_j\) by Claim 11. Since \(UNE[Pr_X^k(C); P] = x_j\), “backward induction” implies \(x_j M^P_{X} x_j\).

Since \(UNE[Pr_X^k; P] = x_1\), “backward induction” implies \(UNE[Pr_X^k(B_1); P] = x_1\) where \(B_1 \equiv \{x_1, x_{j-1}\} \cup X_j^m\) is the “left” successor of the root \(X\) in \(Pr_X^k\). Since the agenda at the “left” successors \(B' \equiv x_1 \cup X_j^m\) of \(B\) and \(B_j\) coincide and \(UNE[Pr_X^k(B_1); P] = x_1\), “backward induction” implies \(UNE[Pr_X^k(B'); P] = UNE[Pr_X^k(B'); P] = x_1\). Since \(UNE[Pr_X(B); P] = x_j\) however, “backward induction” implies that \(x_j M^P_{X} x_j\). Since this contradicts the inference \(x_j M^P_{X} x_j\) drawn in the last paragraph, \(UNE[Pr_X; P] = x_j\).

(iv) This case is similar to (iii) but more involved. Since \(UNE[Pr_X; P] = x_2\) and \(x_2 \notin B\), “backward induction” implies \(UNE[Pr_X(C); P] = x_2\). Let \(B' \equiv x_2 \cup X_j^m\) and \(C' \equiv X_j^m\) denote the successors of \(C\) in \(Pr_X\). Since \(x_2\) only appears once below \(B'\), it is majority preferred to \(UNE[Pr_X(C'); P]\) and every alternative it meets along the “backward induction” path in \(Pr_X(B')\). Now, consider the re-prioritized agenda \(Pr_X^k\), letting \(B_2 \equiv x_2 \cup X_j^m\) and \(C_2 \equiv x_1 \cup X_j^m\) denote the successors of \(X\). By construction, \(B_2 \equiv x_1 \cup X_j^m\) and \(C_2 \equiv C'\) are the successors of \(C\) in \(Pr_X^k\).

Since every alternative that amends \(x_1\) in \(Pr_X\) also amends \(x_2\), everything that \(x_2\) meets along the “backward induction path” in \(Pr_X(B_2)\) is an alternative that it meets in \(Pr_X(B')\). Since \(x_2\) is majority preferred to all of these alternatives by the first observation in the last paragraph, \(UNE[Pr_X^k(B_2); P] = x_2\). Moreover, \(UNE[Pr_X^k(C_2); P] = UNE[Pr_X(C'); P]\) by the second observation in the last paragraph.

By way of contradiction, suppose \(UNE[Pr_X^k; P] \neq x_2\). Since \(UNE[Pr_X^k(B_2); P] = x_2\) and \(x_2 M^P_{X} UNE[Pr_X^k(C_2); P] = UNE[Pr_X(C'); P]\), “backward induction” implies \(UNE[Pr_X^k; P] = UNE[Pr_X^k(B_2); P]\). Since \(x_2\) is majority preferred to everything that \(x_1\) meets along the “backward induction” path in \(Pr_X(B_2)\), \(UNE[Pr_X^k; P] = UNE[Pr_X^k(B_2); P] = x_1\). Since \(UNE[Pr_X^k(B_2); P] = x_2, x_1 M^P_{X} x_2\).

Since \(UNE[Pr_X^k(B_2); P] = x_1\), the same reasoning as above establishes \(UNE[Pr_X(B); P] = x_1\). Since \(UNE[Pr_X; P] = x_2\) however, \(x_2 M^P_{X} x_1\). Since this contradicts the last paragraph, \(UNE[Pr_X^k; P] = x_2\).

9.10 Proof of Theorem 3 (and Proposition 5)

Definition 15 A sequence \(\langle a \rangle \equiv a_1, \ldots, a_n\) is universal for \(x \in A\) on \((P, A)\) if:

1. every alternative in \(A\) appears exactly once in \(\langle a \rangle\);
2. \(\langle a \rangle\) contains an \(M^P_{X}\)-transitive subsequence \(\langle b \rangle \equiv b_1, \ldots, b_m\) such that \(x = b_1\); and,
3. every alternative \(a_i \in \langle a \rangle \setminus \langle b \rangle\) satisfies the following two conditions:
   (i) \(a_i\) appears before some alternative \(b \in \langle b \rangle\) in the sequence \(\langle a \rangle\); and,
   (ii) \(b_j M^P_{X} b_i\) for the first \(b_j \in \langle b \rangle\) after \(a_i\) in the sequence \(\langle a \rangle\).

Claim 24 A sequence \(\langle a \rangle\) in \(A\) is universal on \((P, A)\) for at most one alternative in \(A\).

Proof. By way of contradiction, suppose \(\langle a \rangle = a_1, \ldots, a_n\) is universal for distinct alternatives \(a_i\) and \(a_j\). Without loss of generality, suppose \(i < j\). Let \(\langle b' \rangle\) and \(\langle b'' \rangle\) denote the transitive subsequences associated with \(a_i\) and \(a_j\), respectively.
By condition (3.i), \( b'_1 = a_i \) and \( b'_1 = a_j \). So, \( a_i \notin \langle b' \rangle \). Hence, \( a_j M^\alpha a_i \) by condition (3.ii). Since \( \langle b' \rangle \) is a transitive sequence with \( a_i \), maximal by condition (2), \( a_j M^\alpha a_i \) implies \( a_j \notin \langle b' \rangle \).

By condition (3.ii), \( a_i \notin \langle b' \rangle \) implies \( b'_i M^\alpha a_j \) for some \( b'_i \in \langle b' \rangle \) such that \( b'_i = a_{i(j)} \) and \( k(i) > j \). Since \( \langle b' \rangle \) is a transitive subsequence by condition (2), \( b'_i M^\alpha a_j \) implies \( b'_i \notin \langle b' \rangle \). By condition (3.ii), \( b'_i \notin \langle b' \rangle \) implies \( b'_i M^\alpha b'_i \) for some \( b'_i \in \langle b' \rangle \) such that \( b'_i = a_{i(j)} \) and \( k(j) > k(i) \). Since \( \langle b' \rangle \) is a transitive subsequence by condition (2), \( b'_i M^\alpha b'_i \) implies \( b'_i \notin \langle b' \rangle \). Continuing in this vein leads to the contradiction that \( (a) \) does not terminate.}

**Claim 25** There exists a universal sequence for \( x \) on \((P, A)\) iff \( x \in BA(M^\alpha a)\).

**Proof.** (\( \Rightarrow \)) Fix some \( y \in A \setminus BA(M^\alpha a) \). By way of contradiction, suppose \( (a) \) is universal for \( y \) on \((P, A)\). Consider the transitive subsequence \( \langle b \rangle = b_1, ..., b_m \) with \( b_1 = y \). Since \( y \in A \setminus BA(M^\alpha a) \), \( \langle b \rangle \) is not a maximal transitive sequence. In other words, there exists some \( a \in A \setminus \langle b \rangle \) such that \( a, b_1, ..., b_m \) is a transitive sequence. So, there exists no \( b_i \in \langle b \rangle \) such that \( b_j M^\alpha a \). Since a new appear before some \( b_j \in \langle b \rangle \) by condition (3.i), this contradicts condition (3.ii). In particular, no \( b_j \in \langle b \rangle \) may come after \( a \) in \( (a) \). (\( \Leftarrow \)) Since \( x \in BA(M^\alpha a) \), there exists a maximal transitive sequence \( \langle b \rangle \) in \( A \) with \( b_1 = x \). For every \( a \in A \setminus \langle b \rangle \), there exists some \( b \) such that \( b M^\alpha a \). Otherwise, \( a, b_1, ..., b_m \) is a transitive sequence, which contradicts the maximality of \( \langle b \rangle \). Let \( b(a) \) denote the last alternative in \( \langle b \rangle \) such that \( b(a) M^\alpha a \). Extend \( \langle b \rangle \) into a sequence \( \langle a \rangle \) on \( A \) by inserting each \( a \in A \setminus \langle b \rangle \) between the alternatives \( b(a)_{n+1} \) and \( b(a) \). By construction, \( (a) \) satisfies conditions (1)-(3). So, \( (a) \) is a universal sequence for \( x \) on \((P, A)\). ■

Given a priority agenda \( \Pr^A \) on \( A \) defined by \( \prec, a \), define a strict ordering \( \succ^a \) from \( \succ \). Let \( (a)^* = a_1, ..., a_n^* \) denote the sequence defined by taking the alternatives in \( A \) in the order associated with \( \succ^a \). Given another sequence \( (a) = a_1, ..., a_n \) of the alternatives in \( A \), define the permutation \( \sigma_{(a)} : A \rightarrow A \) by \( \sigma_{(a)}(a_i) = a_i^* \). To simplify the notation, let \( \Pr^A_{\sigma_{(a)}} \) denote the agenda where the \( i \)th alternative in the sequence \( (a) \) occupies the \( i \)th highest priority position in the agenda.

**Claim 26** Given a priority agenda \( \Pr^A \) on \( A \) and a universal sequence \( (a) \) for \( x \) on \((P, A)\), \( UNE[\Pr^A_{\sigma_{(a)}}; P] = x \).

**Proof.** The proof is by strong induction on \(|A|\). The base case \(|A| = 2\) is trivial. To complete the induction, suppose the claim holds for \(|A| = n\) and consider the case \(|A| = n + 1\). Fix a universal sequence \( (a) \) for \( x \) on \((P, A)\). By Claim 24, \( (a) \) is not universal for any other alternative on \((P, A)\). By definition, \( (a) \) contains a transitive subsequence \( (b) \) satisfying conditions (2)-(3) with \( x = b_1 \). Given Claim 21, let \( C \equiv a_1 \cup A^\alpha k+1 \) and \( C' \equiv A^\alpha k+1 \) for \( 2 < k < n + 2 \) denote the successors of the root node in \( \Pr^A_{\sigma_{(a)}} \); and, let \( (c) \) and \( (c') \) denote the subsequences of \( (a) \) associated with \( C \) and \( C' \).

There are three cases to consider for \( x = b_1 \): (i) \( x \in C \setminus C' \); (ii) \( x \in C \cap C' \); and, (iii) \( x \notin C \cap C' \).

(i) Since \( C \setminus C' = \{a_1\}, x = a_1 \). It is easy to verify that \( (c) = x, a_2, ..., a_n^* \) is universal for \( x \) on \((P, C)\). Since the last alternative in \( (a) \) must be in \( (b) \) by condition (3.i), it is also easy to verify that \( (c') = a_2, ..., b_2, ..., a_n^* \) is universal for \( b_2 \) on \((P, C')\). Since \( \Pr^A_{\sigma_{(a)}}(C) \) and \( \Pr^A_{\sigma_{(a)}}(C') \) are both priority agendas, \( UNE[\Pr^A_{\sigma_{(a)}}(C); P] = x \) and \( UNE[\Pr^A_{\sigma_{(a)}}(C); P] = b_2 \) by the induction hypothesis. Since \( x \equiv a_2 b_2 \) by construction, \( UNE[\Pr^A_{\sigma_{(a)}}; P] = x \).

(ii) Since \( C \cap C' = A^\alpha \), \( x = a_j \) for some \( j \geq k \). As in case (i), it is easy to verify that \( (c) = a_1, a_k, ..., x, ..., a_n^* \) is universal for \( x \) on \((P, C)\). Likewise, it is easy to verify that \( (c') = a_2, ..., x, ..., a_n^* \) is universal for \( x \) on \((P, C')\). So, \( UNE[\Pr^A_{\sigma_{(a)}}(C); P] = x = UNE[\Pr^A_{\sigma_{(a)}}(C'); P] \) by the induction hypothesis, which implies \( UNE[\Pr^A_{\sigma_{(a)}}; P] = x \).

(iii) Since \( C' \setminus C = A^\alpha k+1 \), \( x = a_j \) for some \( 2 \leq j < k \). As in case (ii), it is easy to verify that \( (c') = a_2, ..., x, ..., a_n^* \) is universal for \( x \) on \((P, C')\). So, \( UNE[\Pr^A_{\sigma_{(a)}}(C'); P] = x \) by the induction hypothesis. By way of contradiction, suppose \( UNE[\Pr^A_{\sigma_{(a)}}; P] \neq x \). Since \( C \setminus C' = \{a_1\} \), Claim 11 implies \( UNE[\Pr^A_{\sigma_{(a)}}; P] = a_1 \). So, \( UNE[\Pr^A_{\sigma_{(a)}}(C); P] = a_1 \). Since \( UNE[\Pr^A_{\sigma_{(a)}}(C'); P] = x \), it follows that \( a_1 M^\alpha x \), which contradicts condition (3.iii). ■
**Proof of Proposition 5.** Fix a priority agenda \( \text{Pr}_A \) and a profile \( P \) on \( A \). (i') By McKelvey and Niemi’s [1978] observation that no agenda can select an alternative outside the Top Cycle of \( (P, A) \), it follows that \( V_{\Sigma}(\text{Pr}_A; P) \subseteq TC(M_P^A) \). (ii') Consider an \( x \in BA(M_P^A) \). By Claim 25, there exists a universal sequence \( \langle a \rangle \) for \( x \) on \( (P, A) \). Then, \( UNE[\text{Pr}_A^{\langle a \rangle}; P] = x \) by Claim 26. So, \( x \in V_{\Sigma}(\text{Pr}_A; P) \). Equivalently, \( BA(M_P^A) \subseteq V_{\Sigma}(\text{Pr}_A; P) \).

**Claim 27** (i) If \( v \) satisfies IS and ILA and each \( A/ \in X_1 \cup X_2 \) has two marginal alternatives, then \( S_v^X \) is Euro-Latin. (ii) If \( v \) satisfies IS and ILA and each \( A/ \in X_1 \cup X_2 \) has one marginal alternative, then \( S_v^X \) is Anglo-American.

**Proof.** (i) Consider the splitting \((B_1, X_1)\) of \( X \) and let \( a^* \) denote a marginal alternative in \( X \). By Claim 16(i), there are two potential splittings: (1) \( a^* \in B_1 \cap X_1 \) with \( b_1 \in B_1 \setminus X_1 \) and \( x_1 \in X_1 \setminus B_1 \); and, (2) \( (B_1, X_1) = (b_1, X \setminus b_1) \).

For (1), Claim 6 implies that \( \{a^*, b_1, x_1\} \) has one marginal alternative \( a^* \), a contradiction. So, the splitting must be (2). Continuing in the same vein on \( X \setminus c_1 \) and \( X \setminus b_1 \) establishes that \( S_v^X \) is an Anglo-American agenda.

(ii) Consider the splitting \((B, C)\) of \( X \). If \( |C \setminus B| \geq 1 \) (with \( b \in B \setminus C \) and \( c, c' \in C \setminus B \)), then Claim 6 implies that \( \{b, c, c'\} \) has two marginal alternatives \( c \) and \( c' \), a contradiction. This shows that \( |C \setminus B| = |B \setminus C| = 1 \). In other words, \((B, C) = (X \setminus c_1, X \setminus b_1)\) for some \( b_1 \in B \) and \( c_1 \in C \). Continuing in the same vein on \( X \setminus c_1 \) and \( X \setminus b_1 \) establishes that \( S_v^X \) is an Anglo-American agenda.

**Proof of Theorem 3.** Sufficiency: By Claim 27(i) above (resp. Claim 27(ii) above). Necessity: Theorem 1 ensures that every decision rule implementable by Euro-Latin agenda (resp. Anglo-American agenda) satisfies IS and ILA. In turn, it is easy to check that every decision rule implementable by Euro-Latin agenda (resp. Anglo-American agenda) has two marginal alternatives (resp. a unique marginal alternative) for all \( A \) such that \( |A| \neq 1 \).
10 Supplementary Appendix

10.1 Examples

Example 10 In Congress, a **two-stage amendment agenda** arises when a substitute to the original bill (or an amendment to the bill) is proposed. In that case, procedure requires voting to determine the features of the "original" and "substitute" versions of the legislation before deciding whether either of these versions should replace the status quo. When there is a bill $b$, a substitute $b'$, and amendments ($a$ and $a'$) to each, for example, the sequence of questions is:

- **Q1.** Should the amendment $a$ to the original bill $b$ be accepted?
- **Q2.** Should the amendment $a'$ to the substitute bill $b'$ be accepted?
- **Q3.** Should the substitute bill (as amended) be selected over the original bill (as amended)?
- **Q4.** Should the surviving bill be selected over the status quo?

This leads to the agenda below where: $x_1$ (resp. $x_3$) represents $b$ (resp. $b'$); $x_2$ (resp. $x_4$) represents $b + a$ (resp. $b' + a'$); and, $x_5$ represents the status quo (see also Schwartz [2008, Figure 2]; and, Miller [1995, Figure 4]).

Example 11 The following agendas show that the two structural features of recursive agendas are independent.

- **Continuous and repetitive:** Consider the right-hand agenda in Figure 5.
- **Discontinuous and non-repetitive:** Consider the two-stage amendment agenda depicted in Example 10.

Example 12 Up to permutation of the alternatives, there are fifteen recursive agendas for $|X| = 4$: the nine simple agendas (see Example 3); the two agendas in Example 7; and, the four agendas depicted below.
Example 13  The following example show that recursive agendas can violate one or both properties of simple agendas.

- **History-dependent and persistent**: Following Remark 4, consider the left-hand agenda in Figure 5.

- **History-independent and non-persistent**: Consider the left-hand agenda in Example 7. It is easy to check that this agenda is recursive. Since the labels of the non-terminal nodes are distinct, it is also history-independent. However, it is not persistent: no node between the root \( r \) and the terminal node labeled \( x_2 \) has a successor labeled \( u(r) = x_4 \).

- **History-dependent and non-persistent**: Starting with the left-hand agenda in Example 7, replace the left (resp. right) terminal node labeled \( x_4 \) with the agenda under the left (resp. right) node labeled \( \{x, b, c\} \) in Figure 5. The resulting agenda on six alternatives is recursive; but, it clearly violates both properties of simple agendas.

Example 14  The right-hand agenda in Example 1 admits "strict" and "weak" priority representations:
Although \( x_1 \) and \( x_2 \) are not symmetric alternatives, they have equal priority in the “weak” representation.

Example 15 Consider the following extension of the right-hand agenda in Example 1:

\[
\begin{align*}
\{x_1, x_2, x_3, x_4, x_5\} & \quad \{x_1, x_3, x_4, x_5\} & \quad \{x_2, x_3, x_4, x_5\} \\
\{x_1, x_5\} & \quad \{x_3, x_5\} & \quad \{x_2, x_4, x_5\} \\
& \quad \{x_3, x_5\} & \quad \{x_3, x_5\} \\
& \quad \{x_4, x_5\} & \quad \{x_4, x_5\}
\end{align*}
\]

It is easy to see that this agenda is a priority agenda: starting from the weak representation in Example 14, simply define \( x_4 \succ x_5 \) and \( \alpha(x_5) = x_1 \). However, there is no way to define both \( \alpha(x_4) \) and \( \alpha(x_5) \) when one assigns a strict priority between \( x_1 \) and \( x_2 \). The problem is that \( x_4 \) does not amend \( x_1 \); and, \( x_5 \) does not amend \( x_2 \).

10.2 Independence of the Axioms

- IS and ILA but not M: The decision rule on \( X \) induced by the knockout agenda in Example 2 clearly satisfies IS and ILA. To see that it violates M, note that \( x_3 \) is the outcome for any profile \( P \) that gives the majority relation in Figure 9 below; but, \( x_3 \) is not the Condorcet winner on \((P, X)\). By symmetry, there are profiles where each of the alternatives is selected without being the Condorcet winner.

- ILA and M but not IS: The decision rule on \( A \) induced by the right-hand agenda \( T_{X|A} \) in Figure 5 trivially satisfies ILA. To see that it satisfies M, note that \( UNE[T_{X|A}; P] = x \) only if \( x \) is the Condorcet winner on \( A \). As shown in Example 5 however, it violates IS (because there is no way to split \( A \)).

- IS and M but not ILA: Consider the decision rule \( v_3 \) on \( \{x_1, x_2, x_3\} \) that selects: the majority preferred alternative between \( x_1 \) and \( x_2 \) when both are available; \( x_i \) on \( \{x_i, x_3\} \); and, \( x_i \) on \( \{x_i\} \). Since \((\{x_1, x_3\}, \{x_2, x_3\})\) splits \( \{x_1, x_2, x_3\} \),
\(v_3\) satisfies IS. Since \(x_3\) is trivially marginal, \(v_3\) also satisfies M. To see that it violates ILA, consider a profile \(P_3\) where \(x_3\) is the Condorcet winner on \(\{x_1, x_2, x_3\}\). If \(v_3\) satisfies ILA, then \(v_3(P_3, \{x_1, x_2, x_3\}) = \ldots = x_3\). But, this contradicts the assumption that \(v_3(P_3, \{x_1, x_2, x_3\}) \neq x_3\).

10.3 Connection to Apesteguia et al. [2014]

Subsection (a) describes the formal connection while subsection (b) establishes the proofs.

(a) Discussion

The following stipulates that a decision rule marginalizes the same number of alternatives for all issues of a given size.

**Neutral Marginalization (NM)** For each \(j \leq |X|\), \(v\) marginalizes the same number of alternatives for all \(A \in X_j\).

This property is sufficient to distinguish Euro-Latin and Anglo-American agendas from all other priority agendas:

**Proposition 6** Among the decision rules implementable by priority agenda, the only ones that satisfy Neutral Marginalization are the decision rules implementable by Euro-Latin agenda or Anglo-American agenda.

Given Theorem 2, this provides a joint characterization for Euro-Latin and Anglo-American agendas: *a decision rule \(v\) is implementable by one of these agendas if and only if it satisfies IS, ILA, M, and NM. For separate characterizations, recall that each of these agenda types marginalizes the same number of alternatives for every issue of three or more alternatives (see Section 6.2 of the paper). Given Proposition 6, Theorem 3 then follows immediately.*

Apesteguia et al. also give separate characterizations of implementation by Euro-Latin and Anglo-American agenda. Unlike Theorem 3, their approach emphasizes the differences between the resulting decision rules rather than the fundamental similarities. Specifically, they rely on the following two properties:

**Condorcet Priority (CP)** For each \(A \notin X_1 \cup X_2\), there exists a prioritarian alternative \(p^* \in A\) such that, for every Condorcet triple \(P_{p^*xy}\) involving alternatives \(x, y \in A\), \(v(P_{p^*xy}, \{p^*, x, y\}) = p^*\).

**Condorcet Anti-Priority (CA)** For each \(A \notin X_1 \cup X_2\), there exists an anti-prioritarian alternative \(p_* \in A\) such that, for every Condorcet triple \(P_{p_*xy}\) involving alternatives \(x, y \in A\), \(v(P_{p_*xy}, \{p_*, x, y\}) = y\).

To characterize the Euro-Latin rule, they require CP, ILA, and Division Consistency (DC) (see footnote 21 above). For the Anglo-American rule, they require CA, ILA, and a property called Elimination Consistency (EC).

By Claim 28, CP marginalizes two alternatives for every issue (with three or more alternatives) while CA marginalizes one. Given Corollary 3, this implies that one can replace DC and EC in their characterizations by IS:

**Corollary 2** A decision rule \(v\) is implementable by:

(i) a Euro-Latin agenda if and only if it satisfies IS, ILA, and CP;

(ii) an Anglo-American agenda if and only if it satisfies IS, ILA, and CA.

Unlike Apesteguia et al., this result emphasizes the close connection between Euro-Latin and Anglo-American agendas.

(b) Proofs
Proof of Proposition 6. For the necessity of NM: See the necessity proof of Corollary 3.

For the sufficiency of NM: By Theorem 2, every decision rule implementable by priority agenda has a marginal alternative on X. By Claim 15, there are two cases: (i) X has two marginal alternatives (denoted $a_1^*$ and $a_2^*$); or, (ii) X has a unique marginal alternative. I show that $S_X^v$ is Euro-Latin for (i) and Anglo-American for (ii).

(i) By Claim 27(i), it suffices to establish that all A such that $|A| 
eq 1$ have two marginal alternatives. By Claim 14, $a_1^*$ and $a_2^*$ are marginal for $X \setminus x$ for all $x \neq a_1^*, a_2^*$. By NM, this implies that $X \setminus y$ has two marginal alternatives for all $y \in X$. Continuing in the same vein, the desired result follows by a simple inductive argument.

(ii) By Claim 27(ii), it suffices to establish that all A such that $|A| \neq 2$ have one marginal alternative. By way of contradiction, suppose $|X| \geq 4$ and some $X \setminus x$ has two marginal alternatives. Then, by the argument in case (i), every $A \neq X$ such that $|A| \neq 1$ has two marginal alternatives. To establish the contradiction, consider the splitting $(B_1, X_1)$ of X. By the argument in Claim 27(i), $S_X^{X_1}$ is Euro-Latin and $(B_1, X_1) = (b_1, X \setminus b_1)$. This establishes that $S_X^v$ is also Euro-Latin. It then follows that X has two marginal alternatives, which is a contradiction. ■

Claim 28 Suppose v satisfies IS and ILA. (i) If v satisfies CP, then every $A \notin X_1 \cup X_2$ has two marginal alternatives. (ii) If v satisfies CA, then every $A \notin X_1 \cup X_2$ has a unique marginal alternative (the unique anti-prioritarian alternative).

Proof. The proof of (i) [resp. (ii)] is by induction on $|X|$. For $|X| = 3$, the claim follows from ILA and CP [resp. CA]. For the induction step, note that all $A \subset X$ satisfy the claim by the induction hypothesis. To see that X also satisfies the claim, consider the splitting $(B, C)$ of X. There are two possibilities for the prioritarian [resp. anti-prioritarian] p in X: (a) $p = B \cap C$; and, (b) $p = B \setminus C$. I address these possibilities separately for (i) and (ii).

(i) Consider $b \in B \setminus C$ and $c \in C \setminus B$. By Claim 6, (a) leads to the contradiction that p is not prioritarian in \{b, c, p\} (let alone X). So, (b) must hold. Using the same kind of reasoning, it can be shown that $B = \{p\}$. (The idea is to suppose that there exists some $b' \in B \setminus p$. Then, consider an issue \{b', c, p\} such that $c \in C \setminus B$. While there are several cases, a contradiction obtains for each.) By the induction hypothesis, $X \setminus p$ has two marginal alternatives. Since the splitting of X is $(p, X \setminus p)$, IS implies that these alternatives are marginal for X as well.

(ii) Consider $b \in B$ and $c \in C \setminus B$. By Claim 6, (b) leads to the contradiction that p is not anti-prioritarian in \{b, c, p\} (let alone X). So, (a) must hold. By the induction hypothesis, p is marginal for B and C (since it is anti-prioritarian for these issues). By IS, it then follows that p is marginal for X. Finally, by Claim 14 and the induction hypothesis, there can be no other marginal alternative in X. ■

Proof of Corollary 2. For the class of Euro-Latin (resp. Anglo-American) agendas: (⇒) IS and ILA are necessary by Theorem 2; and, CP (resp. CA) is necessary by Theorem 1 (resp. 2) of Apesteguia et al. (⇐) By Claims 27 and 28. ■

10.4 Applications and Extensions

Proof of Remark 6. (Necessity) Theorem 1 establishes the necessity of ILA; and, the necessity of Complete IS is obvious. (Sufficiency) By Complete IS, $S_X^v$ is a knockout agenda. And, by Theorem 1, $S_X^v$ implements v. ■

Proof of Remark 7. (i) This follows directly from the discussion in the text.

(ii) By (the first part of) Weak* Marginalization, there exists a marginal alternative $x^*$ on X. By Claim 14, $x^*$ is marginal on all $A \supset \{x^*\}$. Now, fix an $A \notin X_1 \cup X_2$ such that $A \supset \{x^*\}$ and let $(B, C)$ denote its splitting. By Claim 16(i): (1) $x^* \in B \cap C$ with $b \in B \setminus C$ and $c \in C \setminus B$; or, (2) $(B, C) = (b, X \setminus b)$ with $x^* \neq b$.

By way of contradiction, suppose that (2) obtains. By Claim 6(ii), $(b, \{x^*, c\})$ splits $D = \{b, c, x^*\}$ for some $c \in A$. So, $x^*$ and c
are marginal on $D$ – which contradicts (the second part of) Weak* Marginalization; and, establishes that $x^* \in B \cap C$ for each $A \notin X_1 \cup X_2$ such that $A \supset \{x^*\}$. By construction, it then follows that $S_{X_i}^\prime$ is a standard agenda (with $x^*$ acting as the status quo). Since $v$ satisfies IS and ILA, Theorem 1 further implies that $S_{X_i}^\prime$ is simple and implements $v$. ■

Proof of Remark 8. The agenda $A$ in Example 10 is discontinuous. To see that it satisfies Preference Monotonicity, first observe that the two sub-agendas $A'$ and $A''$ below the root node are simple. So, $A'$ and $A''$ satisfy Preference Monotonicity by Proposition 1. Letting $UNE[A; P] = x_i$ for $i \in \{1, ..., 5\}$, there are five cases to consider:

$i = 1, 2$: Without loss of generality, let $i = 1$ (since the reasoning for $i = 2$ is similar). Since $x_1$ only appears in one sub-agenda at the root (say $A'$), $UNE[A'; P] = x_1$. By Preference Monotonicity of $A'$, $UNE[A'; P^{x_1}] = x_1$. Since $UNE[A''; P] = UNE[A''; P^{x_1}]$ as well, it then follows that $UNE[A; P^{x_1}] = x_1$.

$i = 3, 4$: Without loss of generality, let $i = 3$ (since the reasoning for $i = 4$ is similar). Since $x_3$ appears in both sub-agendas at the root, there are two cases: (i) $UNE[A'; P] = x_3 = UNE[A''; P]$; and, (ii) $UNE[A'; P] = x_3 \neq UNE[A''; P]$.

(i) By Preference Monotonicity of $A'$ and $A''$, $UNE[A'; P^{x_1}] = x_3 = UNE[A''; P^{x_1}]$. So, $UNE[A; P^{x_1}] = x_3$.

(ii) Note that $UNE[A''(\{x_1, x_3, x_5\}); P] = x_3$ so that $x_3 M_{x_3}^{x_3}$. Since $A''$ is simple, $UNE[A''; P] = x_2$ (resp. $x_5$) implies $UNE[A''(\{x_2, x_3, x_5\}); P] = x_2$ (resp. $x_5$) by Claim 11. So, $UNE[A''; P] = x_4$. (If $UNE[A''; P] = x_2$, then $UNE[A''(\{x_2, x_3, x_5\}); P] = x_2$ so that $x_2 M_{x_2}^{x_2}$, which contradicts $UNE[A; P] = x_3$. If $UNE[A''; P] = x_5$, then $x_5 M_{x_5}^{x_5}$, which is a contradiction.)

Since $UNE[A''; P] = x_4$, $UNE[A''(\{x_2, x_4, x_5\}); P] = x_4$. In turn, this implies $UNE[A''(\{x_2, x_3, x_5\}); P] = x_2$ (since $x_2 M_{x_2}^{x_2}$ and $x_3 M_{x_3}^{x_3}$). So, $x_2 M_{x_2}^{x_2}$ and $x_3 M_{x_3}^{x_3}$. Consequently, $UNE[A''(\{x_2, x_3, x_5\}); P^{x_1}] = x_3$ (since $x_3 M_{x_3}^{x_3}$); and, $UNE[A''; P^{x_1}] = x_3$ (since $UNE[A''(\{x_2, x_4, x_5\}); P^{x_1}] = x_4$ and $x_3 M_{x_3}^{x_3}$). Since $UNE[A'; P^{x_1}] = x_3$ by Preference Monotonicity of $A'$, $UNE[A; P^{x_1}] = x_3$.

$i = 5$: Since $x_5$ is in every terminal sub-game, it must be the Condorcet winner on $P$. So, $UNE[A; P^{x_1}] = x_5$. ■

Given a tournament $M_A$, an alternative $a$ covers another alternative $b$ if: $a M_A b$; and, for all $c \in A \setminus \{a, b\}$, $b M_A c$ implies $a M_A c$. In turn, a subset $B \subseteq A$ is $A$-uncovered if, for each $a \in A \setminus B$, there exists a $b \in B$ such that $a$ does not cover $b$.

Definition 16 The Candidate Stable Set $CS(M_A)$ is the subset of alternatives in $A$ that are at the top of some uncovered $M_A$-transitive chain. Formally, $CS(M_A) \equiv \{x \in A : x = b_1 \text{ for some } A \text{-uncovered } M_A \text{-transitive chain } \langle b \rangle \}$.

It is clear that a maximal $M_A$-transitive chain $\langle b \rangle$ must be $A$-uncovered: for each $a \in A \setminus \{b\}$, there exists a $b \in \langle b \rangle$ such that $b M_A a$. This shows that $CS(M_A) \supseteq BA(M_A)$. Using the next definition, one can show much more:

Definition 17 The $k$-Dominating Set $D^k(M_A)$ is the subset of alternatives in $A$ that beat every other alternative in $k$ steps or less. Formally, $D^k(M_A) \equiv \{x \in A : x = b_1 M_A ... M_A b_{n+1} = a \text{ with } n \leq k \text{ for each } a \in A \setminus x \}$.

The tournament solution associated with $k = 2$ is called the Uncovered Set (i.e., $UC(M_A) \equiv D^2(M_A)$ for all $A$ and every tournament $M_A$); and, the tournament solution associated with $k = |A| - 1$ is the Top Cycle. Proposition 6 of Dutta et al. [2002] establishes that $UC(M_A) \subseteq CS(M_A) \subseteq D^3(M_A)$ for all $A$ and every tournament $M_A$.

Proof of Remark 9. Dutta et al. [2002] establish the following results: $\overline{V}_X(AA_A; P) = CS(M_A^*)$ for every profile $(P, A)$ and Anglo-American agenda $AA_A$ (Proposition 5); and, $\overline{V}_X(T_A; P) \subseteq TC(M_A^*)$ for every profile $(P, A)$ and agenda $T_A$ (Proposition 3). To complete the proof, it suffices to show that: (a) $CS(M_A^*) \subseteq \overline{V}_X(P_A; P)$ for every profile $(P, A)$ and priority agenda $P_A$; and, (b) $TC(M_A^*) \subseteq \overline{V}_X(EL_A; P)$ for every profile $(P, A)$ and Euro-Latin agenda $EL_A$. 55
(a) Fix \( x \in CS(M^A_n) \). Let \( \langle b \rangle \) denote the associated \( A \)-uncovered \( M_A \)-transitive chain; and, \( B \) the set of alternatives in \( \langle b \rangle \). Define \( B^- \equiv \{ x \in A \setminus B : bM_{Ax} \text{ for some } b \in B \} \). One can use \( \langle b \rangle \) to construct a universal sequence \( \langle a^- \rangle \) for \( x \) on \( A^- \equiv B \cup B^- \) by inserting each \( a \in B^- \) immediately before the last \( b_i \in \langle b \rangle \) such that \( b_iM_{Aa} \). To extend \( \langle a^- \rangle \) into a sequence \( \langle a \rangle \) on \( A \), insert each \( a \in A^+ \equiv A \setminus A^- \) at the end of \( \langle a^- \rangle \).\footnote{It is worth pointing out that this sequence is different from the one Dutta et al. define in their proof of Proposition 5.} By Claim 26 (and the construction of priority agendas), it follows that \( UNE[P^{\langle a \rangle}_{A^+}; P] = UNE[P^{\langle a^- \rangle}_{A^-}; P] = x \).

To establish the result, consider a candidate profile \( P^c \) such that, for each \( c \in A \setminus x \), \( x \succ c \) for all \( y \in A \setminus c \). I show that \( A^- \) is a Nash equilibrium of the first-stage entry game \((P^{\langle a \rangle}_{A^+}, P, P^c)\). In other words, \( x \in EQ[P^{\langle a \rangle}_{A^+}, P, P^c] \) so that \( x \in V(M^A_n) \) as required. There are two cases: (i) \( c \in A^- \); and, (ii) \( c \in A^+ \). (i) Since \( c = x \) has no incentive to exit, suppose \( c \in A^+ \). Since \( \langle a^- \rangle \) is universal for \( x \) on \( A^- \setminus c \), Theorem 2 and Claim 26 imply that \( UNE[P^{\langle a \rangle}_{A^- \setminus c}; P] = x \).

So, \( c \in A^- \setminus x \) has no incentive to exit. (ii) By construction, \( c M^n_A b \) for all \( b \in B \). So, by definition of \( CS(M^A_n) \), there exists some \( b^- \in B^- \) such that \( b^- M^n_A c \). Since \( c \) has lowest priority on \((P, A^- \cup c)\) but is not the Condorcet winner, Theorem 2 implies that \( UNE[P^{\langle a \rangle}_{A^+ \cup c}; P] \neq c \). So, \( c \in A^+ \) has no incentive to enter given \( P^c \).

(b) Fix \( x \in TC(M^A_n) \). Then, there is a sequence \( \langle a \rangle = a_1, \ldots, a_n \) such that \( a_1 = x \) and \( a_i M^n_A a_{i+1} \) for all \( 1 \leq i \leq n \).

Clearly, \( UNE[EL^{\langle a \rangle}_{A^+}; P] = x \). To establish the result, consider a candidate profile \( P^c \) such that, for each \( c \in A \setminus x \), \( x \succ_c y \) for all \( y \in A \setminus c \). Notice that \( UNE[EL^{\langle a \rangle}_{A \setminus c}; P, P^c] \neq c \). So, \( c \) has no incentive to exit given \( P^c \). So, \( A \) is a Nash equilibrium of the first-stage entry game. In other words, \( x \in EQ[EL^{\langle a \rangle}_{A^+}; P, P^c] \) so that \( x \in V(M^A_n) \) as required.

**Proof of Remark 10.** Fix a profile \( P \) on \( X = \{ x_1, x_2, x_3, x_4 \} \) that induces the following majority relation:

![Figure 9: The majority relation \( M^P_X \) induced by the profile \( P \).](image)

For this profile, the priority agendas (in Example 9) implement the following outcomes up to re-prioritization:

<table>
<thead>
<tr>
<th>( i )</th>
<th>Description</th>
<th>Game Form ( P^X_x )</th>
<th>( V^X(P^X_x; P) )</th>
<th>Tournament Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Euro-Latin (E-L)</td>
<td>Figure 2, left</td>
<td>( X )</td>
<td>Top Cycle</td>
</tr>
<tr>
<td>2</td>
<td>E-L with intermediate amendment</td>
<td>Example 1, left</td>
<td>( X )</td>
<td>Top Cycle</td>
</tr>
<tr>
<td>3</td>
<td>E-L with Anglo-American (A-A) amendment</td>
<td>Figure 3, left</td>
<td>( X \setminus x_4 )</td>
<td>Banks Set</td>
</tr>
<tr>
<td>4</td>
<td>A-A with E-L amendment</td>
<td>Figure 3, right</td>
<td>( X \setminus x_4 )</td>
<td>Banks Set</td>
</tr>
<tr>
<td>5</td>
<td>A-A with intermediate amendment</td>
<td>Example 1, right</td>
<td>( X )</td>
<td>Top Cycle</td>
</tr>
<tr>
<td>6</td>
<td>A-A</td>
<td>Figure 2, right</td>
<td>( X \setminus x_4 )</td>
<td>Banks Set</td>
</tr>
</tbody>
</table>

Table 2: Discrimination by the six priority agendas on \( \{ x_1, x_2, x_3, x_4 \} \).
For every majority relation not equivalent to $M^X_{PR}$ (up to permutation), each agenda in Table 2 can yield any outcome in the Top Cycle. Consequently, $Pr^3_X$, $Pr^4_X$, and $Pr^6_X$ are more discriminating than the other priority agendas.

In terms of bushiness, the priority agendas in Table 2 can be ordered as follows:

\[
\begin{array}{l}
AA_X \rightarrow Pr^5_X \rightarrow Pr^5_X \rightarrow Pr^4_X \\
\downarrow \\
Pr^3_X \rightarrow Pr^2_X \rightarrow Pr^1_X \rightarrow EL_X
\end{array}
\]

Figure 10: Bushiness ordering of the priority agendas on $X$.

This shows that $Pr^5_X$ is at once strictly bushier than $Pr^3_X$ and $Pr^4_X$ and strictly less discriminating. ■

Incidentally, the outcomes associated with the three remaining simple agendas (from Examples 2 and 3) are:

<table>
<thead>
<tr>
<th>$i$</th>
<th>Game Form $S^i_X$</th>
<th>$V^*_X(S^i_X; P)$</th>
<th>Tournament Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Example 2</td>
<td>${x_2, x_3}$</td>
<td>Copeland Set</td>
</tr>
<tr>
<td>2</td>
<td>Example 3, left</td>
<td>$X \setminus x_3$</td>
<td>Banks Set</td>
</tr>
<tr>
<td>3</td>
<td>Example 3, right</td>
<td>$X \setminus x_3$</td>
<td>Banks Set</td>
</tr>
</tbody>
</table>

Table 3: Discrimination by the remaining simple agendas on $\{x_1, x_2, x_3, x_4\}$.

As in the proof of Remark 10, the agendas in Table 3 implement the Top Cycle for any majority relation which is not a four-cycle. So, the knockout agenda $S^1_X$ is the most discriminating simple agenda on four alternatives.

Finally, the outcomes associated with the six remaining recursive agendas (from Examples 7 and 12) are:

<table>
<thead>
<tr>
<th>$i$</th>
<th>Game Form $R^i_X$</th>
<th>$V^*_X(R^i_X; P)$</th>
<th>Tournament Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Example 7, left</td>
<td>$X \setminus x_4$</td>
<td>Banks Set</td>
</tr>
<tr>
<td>2</td>
<td>Example 7, right</td>
<td>$X$</td>
<td>Top Cycle</td>
</tr>
<tr>
<td>3</td>
<td>Example 12, top left</td>
<td>$X \setminus x_4$</td>
<td>Banks Set</td>
</tr>
<tr>
<td>4</td>
<td>Example 12, top right</td>
<td>$X \setminus x_4$</td>
<td>Banks Set</td>
</tr>
<tr>
<td>5</td>
<td>Example 12, bottom left</td>
<td>$X \setminus x_4$</td>
<td>Banks Set</td>
</tr>
<tr>
<td>6</td>
<td>Example 12, bottom right</td>
<td>$X$</td>
<td>Top Cycle</td>
</tr>
</tbody>
</table>

Table 4: Discrimination by the remaining recursive agendas on $\{x_1, x_2, x_3, x_4\}$.

Since the agendas in Table 4 likewise implement the Top Cycle for any majority relation that does not include a four-cycle, the knockout agenda is also the most discriminating recursive agenda on $\{x_1, x_2, x_3, x_4\}$.

**Proof of Remark 11.** Fix a collection of agendas $T$; and, suppose two sub-collections $T_1, T_2 \subseteq T$ satisfy Self-Consistency. Then, clearly $T_1, T_2 \subseteq T_1 \cup T_2 \subseteq T$; and, moreover, $T_1 \cup T_2$ satisfies Self-Consistency. ■